The Wright ω function

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2 Graphs and special values

A graph of *!*(z) for real z can be produced easily in Maple by the command `plot([y+ln(y),y,y=0.001..2]);`. A section of the Riemann surface for *!*(z) can be plotted by the following commands:

```
omega := mu + I*nu;
x := evalc(Re(omega+ln(omega)));
y := evalc(Im(omega+ln(omega)));
plot3d( [x,y,mu], mu=-4..2, nu=-4..4,
        colour=black, axes=BOXED,
```

In addition to this basic point, we here present new branch point series (with the correct closure), new asymptotic series (from the equivalent series for the Lambert W function), and new proofs of the analytic properties of $\omega(z)$, using properties of the unwinding number.

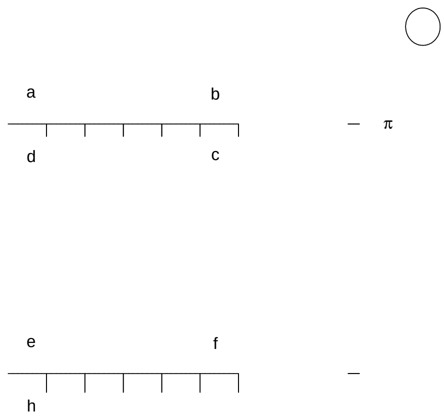

Fig. 1. The z-plane, showing the slit (equivalently, branch cut) we call the “doubling line” (above) and its “reflection”, across each of which the Wright ω function is discontinuous. Along both slits, the closure (indicated by short lines extending down from the slits) is taken from below — clockwise around the branch points — to agree with the closure of the unwinding number.

We here summarize some properties of ω, proved in [9]. First, equation (2) has a unique solution, $\omega(z)$, for all $z \in \mathbb{C}$ except on the line L_D defined by $z = t \pm i\pi$ for $t \le -1$. When z is on L_D, the equation has precisely two solutions, these being $\omega(z)$ and $\omega(z - 2\pi i)$; we therefore call L_D the “doubling line”. See Figure 1 and Figure 2. On the reflection of the doubling line, namely, the line defined by $z = t - i\pi$, with $t \le -1$, equation (2) has no solution at all[2]. Second, ω is an analytic function of z except on the doubling line *and its reflection* $z = t - i\pi$ for $t \le -1$, where $\omega(z)$ is discontinuous. This immediately gives the following.

[2] 3/F1f09.96Tf11.43al7Figl310n364(call)08(hadn'tu.24/l)r6-2callatgivatW=

Theorem: For all $z \in \mathbb{C}$ and integers k,

$$W_k(z) = \omega(\ln_k(z)); \qquad (3)$$

where $\ln_k(z) = \ln z + 2\pi i k$. [This logarithmic notation is discussed further in a later section.]

Proof. This holds at least provided z is not in the interval $-\exp(-1) \le z < 0$ and $k = -1$, which is the image in the domain of W of the critical doubling line (and also the image of its reflection). If z is in the interval $-\exp(-1) \le z < 0$, and $k = -1$, then we have instead that $W_0(z) = \omega(\ln|z| + i\pi)$ since $K(\ln|z| + i\pi) = 0$, and that $W_{-1}(z) = \omega(\ln|z| - i\pi)$ since $K(\ln|z| - i\pi) = -1$. Phrasing this the other way, we have

$$W_0(z) = \omega(\ln z)$$

and

$$W_{-1}(z) = \omega(\ln z - 2\pi i).$$

∎

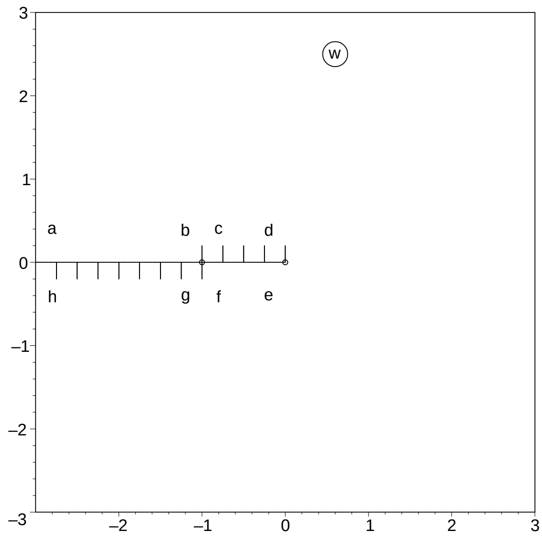

Fig. 2. The ω-plane, showing the images of doubling slit and its reflection. The negative real ω-axis is not, *per se*, a branch cut (this is the range of the function) but it is a branch cut of $\omega + \ln \omega$, which is why that expression is not exactly the inverse function for ω.

2.2 Properties of ω

We group the properties into analytic properties and algebraic properties.

Analytic properties Theorems and lemmas:

(i) $\omega(z)$ is single-valued

(ii) $\omega : \mathbb{C} \to \mathbb{C}$ is onto $\mathbb{C} \setminus \{0\}$.

(ii)(a) Except at $z = -1 \pm i\pi$, where $\omega(z) = -1$, $\omega : \mathbb{C} \to \mathbb{C}$ is injective; hence ω^{-1} exists uniquely except at 0 and -1.

(iii) See Figure 2.

$$\omega^{-1}(y) = \begin{cases} y + \ln(y) - 2\pi i & -\infty < y < -1 \\ -1 \pm i\pi & y = -1 \\ y + \ln(y) & \text{otherwise}. \end{cases}$$

(iv) (a) ω is continuous (in fact analytic) except at $z = t \pm i\pi$ for $t \le -1$.

(b) For $z = t \pm i\pi$ and $t \le -$

(v) (a) $W_{K(z)}(e^z)e^{W_{K(z)}(e^z)} = e^z = \omega(z)e^{\omega(z)}$ by definition. Taking logs, $\ln(\omega e^\omega) = \ln e^z$, or $\omega + \ln\omega - 2\pi i K(\omega + \ln\omega) = z - 2\pi i K(z)$. Therefore, $\omega + \ln\omega = z \iff K(\omega + \ln\omega) = K(z)$.

(v) (b) $K(W_{K(z)}(e^z) + \ln W_{K(z)}(e^z)) = K(z)$. $K(a)$ can change only when $a = t + (2k+1)\pi$ for $k \in \mathbb{Z}$, or when a is itself discontinuous. We distinguish two cases, therefore:

(1) $W_{K(z)}(e^z) + \ln W_{K(z)}(e^z)$ can be discontinuous at discontinuities of $K(z)$, namely $z = t + (2k+1)\pi$ for $k \in \mathbb{Z}$, or when $W_{K(z)}(e^z) < 0$. We ignore discontinuities of $K(z)$ for the moment. $W_{K(z)}(e^z) < 0$ only when (i) $K(z) = 0$ and $e^z < 0 \iff z = t + i\pi$, $t \le -1$, or (ii) $K(z) = -1$ and $e^z < 0 \iff z = t - i\pi$, $t \le -1$. Both (i) and (ii) are discontinuities of $K(z)$ anyway.

(2) $K(\omega(z) + \ln\omega(z))$ can be discontinuous when $\omega + \ln\omega = t + (2k+1)\pi i \Longrightarrow \omega e^\omega = -e^t \iff \omega(z) \in$ an image of $\mathbb{R}^-$ under W. Therefore $z \in$ a pre-image of $\mathbb{R}^-$ under e^z.

But this is just $z = t + (2k+1)\pi i$, which is a place of discontinuity of $K(z)$. Note that K is integral-valued. Therefore, if $\omega(z)$ is such that $K(\omega(z) + \ln(\omega(z))) = K(z)$ for any z in a strip $(2k-1)\pi < \operatorname{Im} z \le (2k+1)\pi$, where $\omega + \ln(\omega)$ is continuous, then we have $K(\omega(z) + \ln\omega(z)) = K(z)$ everywhere in that strip. Let us choose $k \in \mathbb{Z}$, and look at the pre-image of $\omega = 2k\pi i$. Then $\omega + \ln\omega = 2k\pi i + \ln(2k\pi) + i\pi/2$ and hence $K(\omega + \ln\omega) = k$. Since $\omega = W_{K(z)}(e^z)$ we have $\omega e^\omega = e^z$ and $2k\pi i \cdot e^{2k\pi i} = e^z \iff e^z = 2k\pi i$; *moreover* $2k\pi i \in$ range $W_{K(z)}$, and therefore $K(z) = k$. Therefore

$$
\begin{aligned}
z &= \ln(2k\pi i) + 2k\pi i \\
&= \omega + \ln\omega.
\end{aligned}
$$

This establishes that if $\omega(z) = W_{K(z)}(e^z)$, then $\omega + \ln\omega = z$ except possibly on the edges of the strips $z = t + (2k+1)\pi i$. Now we have $K(\omega(z) + \ln(\omega(z))) = K(z)$ if $(2k-1)\pi < \operatorname{Im}(z) < (2k+1)\pi$, and hence $\omega + \ln\omega = z$. Note that $\omega(z) = W_{K(z)}(e^z)$ is continuous from below as $\operatorname{Im}(z) \to (2k+1)\pi^-$. Therefore, provided that $\omega(z) \notin \mathbb{R}^-$, $\omega(z) + \ln(\omega(z))$ will be continuous as $\operatorname{Im}(z) \to (2k+1)\pi^-$. Therefore, since $\operatorname{Im}(\omega(z) + \ln\omega(z)) = \operatorname{Im}(z)$ for $(2k-1)\pi < \operatorname{Im}(z) < (2k+1)\pi$, we have $K(\omega + \ln\omega) = K(z)$ even if $\operatorname{Im}(z) = (2k+1)\pi$ by continuity:

$$
\begin{aligned}
&\lim_{\operatorname{Im}(z) \to (2k+1)\pi^-} K(\omega(z) + \ln\omega(z)) \\
&= \lim_{\operatorname{Im}(z) \to (2k+1)\pi^-} K(z).
\end{aligned}
$$

Therefore $K(\omega(z) + \ln\omega(z)) = K(z)$ unless $\omega(z) < 0$, and $\operatorname{Im}(z) = -i\pi$.

(vi) This now follows immediately.

2.4 Corollary

Define $z(k;\theta) = x + i \cdot (2k + \theta)\pi$. Then $z(k + 1; -1) = z(k; 1)$ since $x + i \cdot (2k + 2 - 1)\pi = x + i \cdot (2k + 1)\pi$, since $K(x + i \cdot (2k + \theta)\pi) = k$ for $-1 < \theta \le 1$. Since

$$W_k(e^{x+i(2k+\theta)\pi}) = W_k(e^{x+i\pi\theta}) = W_k(e^x(\cos \pi\theta + i \sin \pi\theta)) \, ;$$

we have $W_k(e^{x+i\pi\theta}) \to W_k(-e^x + i \cdot 0^+)$ as $\theta \to 1^-$, and

$$\lim_{\theta \to -1^+} W_{K(z(k+1;\theta))}(e^{z(k+1;\theta)}) = \lim_{\theta \to -1^+} W_{k+1}(e^{x+i(2k+2+\theta)\pi})$$

since $K(x+i\cdot(2k+2+\theta)\pi$

– Series about $z = a$, where $a = \omega_a + \ln \omega_a$: the following (computed by Maple) is the beginning of the series for ω which contains second order Eulerian numbers.

$$
\begin{aligned}
&\omega_a + \frac{\omega_a}{1+\omega_a}(z-a) + \frac{1}{2!}\frac{\omega_a}{(1+\omega_a)^3}(z-a)^2 \\
&- \frac{1}{3!}\frac{\omega_a(2\omega_a-1)}{(1+\omega_a)^5}(z-a)^3 + \frac{1}{4!}\frac{\omega_a(6\omega_a^2-8\omega_a+1)}{(1+\omega_a)^7}(z-a)^4 \\
&- \frac{1}{5!}\frac{\omega_a(24\omega_a^3-58\omega_a^2+22\omega_a-1)}{(1+\omega_a)^9}(z-a)^5 \\
&+ \frac{1}{6!}\frac{\omega_a(120\omega_a^4-444\omega_a^3+328\omega_a^2-52\omega_a+1)}{(1+\omega_a)^{11}}(z-a)^6 \\
&+ O((z-a)^7)
\end{aligned}
$$

The general term is [6]:

$$\omega(z) = \sum_{n \ge 0} \frac{q_n(\omega_a)}{(1+\omega_a)^{2n-1}} \frac{(z-a)^n}{n!} \tag{4}$$

where

$$q_n(w) = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1}. \tag{5}$$

is defined in terms of second order Eulerian numbers.

– Series about ∞: This series was originally due to de Bruijn, and Comtet identified the coefficients as Stirling numbers.

$$
\begin{aligned}
&\omega \approx z - \ln(z) + \frac{\ln(z)}{z} + \frac{1}{2}\frac{\ln(z)(\ln(z)-2)}{z^2} \\
&+ \frac{1}{6}\frac{\ln(z)(-9\ln(z)+6+2\ln(z)^2)}{z^3} + \frac{1}{12} \\
&\frac{\ln(z)(3\ln(z)^3-22\ln(z)^2+36\ln(z)-12)}{z^4} + \\
&\frac{1}{60}\ln(z)(-125\ln(z)^3+350\ln(z)^2+12\ln(z)^4 \\
&-300\ln(z)+60)/z^5 + O(\frac{1}{z^6})
\end{aligned}
$$

The general term is (translating from the Lambert W results of [2, 7]) $\omega(z) =$

$$z - \ln z + \sum_{\ell \ge 0}\sum_{m \ge 0} c_{\ell m} \frac{\ln^m z}{z^{\ell+m}} \tag{6}$$

where $c_{\ell m} = (-1)^\ell \left[{\ell+m \atop \ell+1} \right] / m$

can be rearranged in several ways, following [9] and [6]: $\omega(z) =$

$$z - \ln z + \sum_{n\geq 1} \frac{(-1)^n}{z^n} \sum_{m=1}^{n} \frac{(-1)^m}{m!} \left[{n \atop n-m+1} \right] \ln^m z . \tag{7}$$

Using a new variable $\zeta = z/(1+z)$, we get $\omega(z) =$

$$z - \ln z + \sum_{m\geq 1} \frac{\ln^m z}{m! z^m} \sum_{p=0}^{m-1} (-1)^{p+m-1} \zeta^{p+m} \left\{ {p+m-1 \atop p} \right\}_{\geq 2} \tag{8}$$

where the numbers in curly braces are 2-associated Stirling numbers. Using $L_\tau = \ln(1-\tau) = \ln(1-\ln z/z)$ and $\eta = \sigma/(1-\tau) = 1/(z(1-\ln z/z)) = 1/(z-\ln z)$, series (83) and (84) from [6] become

$$\begin{aligned}\omega(z) &= z - \ln z - L_\tau \\ &+ \sum_{n\geq 1} (-\eta)^n \sum_{m=1}^{n} (-1)^m \left[{n \atop n-m+1} \right] \frac{L_\tau^m}{m!}\end{aligned} \tag{9}$$

and

$$\begin{aligned}\omega(z) &= z - \ln z - L_\tau \\ &+ \sum_{m\geq 1} \frac{1}{m!} L_\tau^m \eta^m \sum_{p=0}^{m-1} \left\{ {p+m-1 \atop p} \right\}_{\geq 2} \frac{(-1)^{p+m-1}}{(1+\eta)^{p+m}} .\end{aligned} \tag{10}$$

The series converge for large enough real z

where the double conjugation gives us the correct closure from below on $t + i\pi$ for $t \le -1$. Near $z = -1 - i\pi$,

$$\omega(z) = -\sum_{n \ge 0} a_n \left(-i \overline{\sqrt{\overline{2(z + 1 + i\pi)}}} \right)^n . \qquad (2)$$

In both cases a_n is given by the recurrence relation [10]

$$a_0 = a_1 = 1$$
$$a_k = \frac{1}{(k+1)a_1} \left(a_{k-1} - \sum_{i=2}^{k-1} i a_i a_{k+1-i} \right) . \qquad (3)$$

The derivation of these series from the results of [10] is straightforward, except for the use of $\overline{\sqrt{\overline{z}}}$. We here verify that this construction, which is one of a family of transformations modelled on some used by G.K. Batchelor, gives us the correct closure. We know that $\omega(t + i\pi^-) = W_0(-e^t)$ whilst $\omega(t + i\pi^+) = W_1(-e^t)$, and $\omega(t - i\pi^+) = W_0(-e^t)$ whilst $\omega(t - i\pi^-) = W_{-1}(-e^t)$. Putting $z = t + i\pi^+$ in $\overline{\sqrt{\overline{2(z + 1 - i\pi)}}}$ gives $\overline{\sqrt{\overline{2(t + 1 + i \cdot 0^+)}}}$, for $t \approx -1$. If $t + 1 \ge 0$ then we have no branch cut to cross — this series will be continuous, therefore, along the line $t+1+i\pi$, $t \ge -1$. If $t+1 < 0$, we are on the branch cut, $\overline{t + 1 + i \cdot 0^+}$ is $t+1+i \cdot 0^-$, and arg $\sqrt{2(t + 1 + i \cdot 0^-)} = -\pi/2$. Therefore arg $\overline{\sqrt{\overline{2(t + 1 + i \cdot 0^-)}}} = +\pi/2$, and this means that the series (2) can be written

$$\omega(z) = -\sum_{n \ge 0} a_n (\rho)^n$$

and by inspection of the signs of the series for $W_{-1}(-e^t)$ and hence $W_{+1}(-e^t)$ just above the branch cut, this is correct. [Here $\rho = \sqrt{-2(t + 1)} > 0$.] Next, consider $z = -1 + i\pi^-$. A similar argument leads to the conclusion

$$\omega(z) = -\sum_{n \ge 0} a_n (-\rho)^n$$

which is the series for $W_0(-e^t)$ for $t \approx -1$, because its signs alternate. Consideration of $z = t - i\pi^+$ and $t - i\pi^-$ gives, for $t + 1 < 0$,

$$\begin{aligned} \omega(z) &= -\textstyle\sum_{n \ge 0} a_n (-\rho)^n & z &= t - i\pi^+ \\ &= -\textstyle\sum_{n \ge 0} a_n \rho^n & z &= t - i\pi^- \end{aligned}$$

and continuity if $t + 1 \ge 0$.

Remark. The use of $\overline{\sqrt{\overline{(z - a)}}}$ to represent a square root function with a closure different from the CCC closure, as explained by Kahan, is a useful tool in a computer algebra setting. However, it relies on the designers to be sophisticated enough to provide symbolic means of representing (and not over-simplifying) these series, and the users to be sophisticated enough to know that $\overline{\sqrt{\overline{z}}} \ne \sqrt{z}$ on the branch cut.

3 Interpolating $W_k(z)$

Finally, we interpret equation (3) as an interpolation scheme for $W_k(z)$. We note that k need not be an integer in that equation; the geometric interpretation is precisely that of a circular cylinder cutting the Riemann surface for W. Note also that $k = 0$ and $k = -1$ are special, and not interpolated by this scheme.

We deduce that $W_k(z)$ is, in some sense, analytic in k, except if $-\exp(-1) \le z < 0$ and $k = 0$ or $k = -1$.

$$
\begin{aligned}
\frac{dW_k(z)}{dk} &= \frac{d}{dk}\,\omega(\ln z + 2\pi i k) \\
&= 2\pi i \frac{\omega(\ln z + 2\pi i k)}{1 + \omega(\ln z + 2\pi i k)}.
\end{aligned}
$$

By the analytic properties of ω, this derivative is not continuous on $-\exp(-1) \le z < 0$ at k

the same point, even though these series were all introduced in the same paper [4]. We think that this is because the series are defined *piecewise*: for W_{-1} and W_1

π; this then gives us a *negative imaginary part* on the order of roundoff in the result of the call to exp. This is all explainable in terms of the Maple model of floating-point arithmetic, but it's a disaster nonetheless — one made visible by the next step, the computation of $W_0(x - i\cdot\varepsilon)$, which is *on the wrong side of the branch cut*. The numerical value of $W_0(x - i\cdot\varepsilon)$ is not at all close to the value of $W_0(x + i\cdot\varepsilon)$, and this discontinuity is *spurious*. The ω function is *continuous* at this point. So: we should have a separate routine for the numerical evaluation of ω that guarantees that we get continuity (where ω is continuous), because the definition combines *discontinuous* functions in such a way that their discontinuities (mostly) cancel.

There are other advantages to using the Wright ω function directly.

1. In addition to being single-valued, ω is continuous (indeed analytic) for all z not on the two half-lines $z = t \pm i\pi$ for $t \leq -1$. It is discontinuous across these lines.
2. The Wright ω

2. Discontinuity (along the branch cuts) is especially visible, and nontrivial, in this function. Therefore it will make a good test case for reasoning about complex-valued expressions.
3. The methods used to prove properties of *!* are essentially old-fashioned mathematics, not commonly seen in standard curricula, and may potentially be automated. This is in the spirit of [3] and represents a potentially interesting direction for future research.

Acknowledgements. The code for the numerical evaluation of $!(z)$ (which will be discussed in a future paper) was scrutinized by Dave Hare. The colour graph of the Riemann Surface for *!* (not shown here, but used in the upcoming Lambert *W* poster, was produced with a Maple program written by George Labahn (getting the colours to come out continuous, while making Maple show the discontinuity, is not trivial.) Jon Borwein and Bill Gosper provided motivation to look at *!*. Our thanks also to Prof. John Wright (Reading & Oxford), and his father Sir Edward Maitland Wright, for permission to use a picture of Sir Edward on the poster of the Lambert *W* function.

References

[1] Bronstein, M., and Davenport, J. H. Algebraic properties of the Lambert *W* function.

[2] Comtet, L. *Advanced Combinatorics*. Reidel, 1974.

[3] Corless, R. M., Davenport, J. H., David J. Jeffrey, Litt, G., and Watt, S. M. Reasoning about the elementary functions of complex analysis. In *Proceedings AISC Madrid* (2000), vol. 1930 of *Lecture Notes in AI*, Springer. Ontario Research Centre for Computer Algebra Technical Report TR-00-18, at http://www.orcca.on.ca/TechReports.

[4] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., and Knuth, D. E. On the Lambert *W* function. *Advances in Computational Mathematics 5* (1996), 329{359.

[5] Corless, R. M., and Jeffrey, D. J. The unwinding number. Sigsam *Bulletin 30*, 2 (June 1996), 28{35.

[6] Corless, R. M., Jeffrey, D. J., and Knuth, D. E. A sequence of series for the Lambert *W* function. In *Proceedings of the ACM ISSAC, Maui* (1997), pp. 195{203.

[7] de Bruijn, N. G. *Asymptotic Methods in Analysis*. North-Holland, 1961.

[8] Graham, R. L., Knuth, D. E., and Patashnik, O. *Concrete Mathematics*. Addison-Wesley, 1994.

[9] Jeffrey, D. J., Hare, D. E. G., and Corless, R. M. \Unwinding the branches of the Lambert *W* function". *Mathematical Scientist 21* (1996), 1{7.

[10] Marsaglia, G., and Marsaglia, J. C. \A new derivation of Stirling's approximation to $n!$". *American Mathematical Monthly 97* (1990), 826{829.

[11] Siewert, C. E., and Burniston, E. E. \Exact analytical solutions of $ze^z = a$". *Journal of Mathematical Analysis and Applications 43* (1973), 626{632.

[12] Wright, E. M. \Solution of the equation $ze^z = a$". *Bull. Amer. Math Soc. 65* (1959), 89{93.