It is natural to ask the question in this form, because this is the sort of equation contained in all the algebra textbooks, and all the reference books. The trouble is that when a group of people start to debate such an equation, *every person is correct*. One person will give a de¯nition that makes the statement true, and another will give a de¯nition that makes it false, but as long as each system of de¯nitions and implications is consistent, both parties are correct. As a result, the unpublished discussions we have witnessed often fall into predictable patterns. Each person states \When I was an undergraduate in university X, I was always

2 Cardano solution formulae

```
In[1]= Sqrt[-2]
Out[1]= i Sqrt[2]
In[2]=N[Out[1]]
Out[2]= 0.+1.41421 i
```

| Mathematica, *Personal communication*

`When *I* use a word,' Humpty Dumpty said in rather a scornful tone, `it means just what I choose it to mean | neither more nor less.'

`The question is,' said Alice, `whether you *can* make words mean di®erent things.'

`The question is,' said Humpty Dumpty, `which is to be master | that's all.'
| Lewis Carroll, *Through the looking glass*

It is a mistake to think that square root and cube root do not need to be de¯ned. As an example of how easily confusion can enter, we consider the use of the symbol $\sqrt{\ }$ in *Numerical Recipes* §5.6 [21], where there are uncommented, but implied, changes in the meaning of notation. Thus we ¯rst read that $ax^2 + bx + c = 0$ has the solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} :$$

The fact that the authors write $\pm$ implies that $\sqrt{\ }$ has a de¯nite sign. However, after their equation (5.6.6), we read that \the sign of the square root should be chosen so as to make

$$\mathrm{Re}\ \overset{3}{b}$$

3.3 Principal value: $\sqrt[n]{z}$

This notation is used for the single-valued, principal-branch function [10], de¯ned by

$$\sqrt[n]{z} = |z|^{1/n} \exp\left(\frac{\arg z}{n}\right) . \tag{14}$$

Here, $\arg z \in (-\pi, \pi]$, and we allow the default case $\sqrt{z} = \sqrt[2]{z}$. (Questions regarding branch-cut placement and closure are more examples of interesting red herrings [10, 22].)
Example: $\sqrt[3]{-8} = 1 + i\sqrt{3}$.

3.4 Real branch: $\mathrm{surd}(z, n)$

Maple accommodates users who want the cube root of a negative real to be a negative real by de¯ning a separate function called `surd`. We adopt that notation here. If $x \in \mathbb{R}$, then $\mathrm{surd}(x, 3) = \mathrm{sgn}(x)\sqrt[3]{|x|}$, where the cube root is real. More generally, if $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, then $\mathrm{surd}(z, n$

5 Commentary on the solutions

Some observations can now be made about the above calculations, keeping in mind our primary focus: the question of which of the de¯nitions §3.1 { §3.4 is best for a computer algebra system.

5.1 Using sets of values

From the point of view of implementing this approach in a CAS, the manipulations of the lists of values are not di±cult. An obvious drawback is one of e±ciency. Although we knew from the start that only 3 solutions existed, we still ended up testing 21 candidate solutions. A potential drawback that is not illustrated by the example concerns the selection of correct answers. The correct solutions were easily identi¯ed here, but identifying zero from °oating-point data can potentially be di±cult.

The numerical example also illustrates the di±culties inherent in trying to implement this de¯nition symbolically. We see that the sets for s

6 Riemann surfaces

There is at least one other way of computing with fractional powers, but we have not discussed it here. In conversations at conferences, one sometimes hears the assertion that Riemann surfaces should be used for fractional power computations. We agree that Riemann surfaces form a beautiful theory, and that they are an attractive way to think about multivalued functions. However, what has not been demonstrated, to our knowledge, is the fact that they can be computational. Perhaps some reader can contribute a demonstration of the way in which the theory of Riemann surfaces can be used to solve the example problem discussed here.

7 Concluding remarks

We think that we have demonstrated that principal-valued functions are the best ones for a computer algebra system, and indeed for most mathematical computations, but we hope that this is not the end of the discussion. It would be useful, for example, to have more problems with which the different approaches could be compared. For the present example problem, principal-value functions gave the best solution, but we are open to the possibility that there are other problems for which this conclusion is not correct. Certainly the surd function was introduced into Maple so that users could use it, not to prove that it was useless. This is a moot point, however, since we must ask how many mathematicians, even Maple users, know of the function. We would like to hear about other system perspectives on the problem. Did discussions similar to this one take place during the development of other systems, such as (in chronological order) Macsyma, Derive and Mathematica? How should these ideas be propagated to the rest of the mathematical world?

We see a connection between the present discussion and Buchberger's advocacy of mathematical research in the computer age [6, 7]. In both cases, the algorithmic aspect of mathematics, whether old mathematics or new, is being raised to a higher level — let us call it the algorismation of mathematics[1].

We close with a few memories of the Maple discussions and the reasons advanced for preferring the multivalued interpretation of functions.

- *Stupid Maple*. Some developers were concerned that if Maple refused to honour a request to expand $\ln(AB)$ into $\ln A + \ln B$, then the users would dismiss the product as being ignorant.
- *Ugly Maple*. Some developers pined for the good old days when they could enjoy the simplicity of

$$\left(\left(\left(\left((z^2)^{\frac{1}{2}}\right)^3\right)^{\frac{1}{3}}\right)^4\right)^{\frac{1}{4}} \to z\,.$$

- *Unresponsive Maple*. Sometimes one wants to force a transformation such as $\ln(AB) \to \ln A + \ln B$, because it might prove to be correct later. Equation solving is a typical situation. Sometimes one

Appendix: Properties of Cardano formulae

Although the Cardano formulae were introduced as an example for testing multivalued de¯nitions, and were not the main subject of this article, several reviewers were stimulated to re-examine the properties of (2){(10). Since, to some extent, the main discussion requires that the reader have con¯dence in the formulae, we record some of the reviewers' comments and questions here.

The formulae look the same. One of the problems of working with principal-valued functions in the complex plane is the need to `unlearn' some of our re°exive use of transformations. As the numerical examples show, the forms are not equivalent.

The formulae are discontinuous. Assuming principal-valued functions, one ¯nds that the Cardano expressions are discontinuous, even though the roots of the polynomial vary continuously with the coe±cients. The observation is correct, but it does not invalidate the correctness of the expressions.

Are the expressions always correct? For the purposes of this paper, we are accepting that Maple and Mathematica are using expressions that are correct for principal-branch functions. A full discussion of the correctness and the merits of the di®erent forms of the Cardano formulae must wait for a separate paper.

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