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Unwinding the branches of the Lambert W function

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Abstract

An algebraic relation is derived that allows the different branches of the Lambert W function to be concisely distinguished. The derivation relies on the unwinding number, which is defined here, for the manipulation of elementary functions in the complex plane.

Key words: Lambert W function; unwinding number.
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1. Introduction

Interest in the Lambert W function goes back at least to Lambert and Euler, albeit that these two mathematicians studied it only indirectly, but the function did not receive a permanent name until it was included in the library of Maple, the computer algebra system. After gaining its name, it gained recognition. Many of the applications of W were found because of users discovering that Maple had used W in the solution of a problem they had posed. In addition to its applicability, the function has rich mathematical properties. It is defined to be the solution $W(z)$ of

$$W(z)e^{W(z)} = z \ , \tag{1}$$

where z is a complex variable. The history, applications and properties of W were recently reviewed in Corless *et al.*[3], and so just two examples of the uses of W are given here, in order to convey the flavour of its applications.

A common way to meet W is through the problem of iterated exponentiation, which is the evaluation of

$$h(z) = z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} \ ,$$

whenever it makes sense. The function h

equation $h(z) = z^{h(z)}$ for $h(z)$ (using Maple, for example), and getting

$$h(z) = -\frac{W(-\log z)}{\log z} . \tag{2}$$

For more details and references on the problem, see Baker & Rippon [2].

Another application is the solution of linear constant-coefficient delay equations [8]. Consider the simple delay equation

$$\dot{y}(t) = ay(t-1) , \tag{3}$$

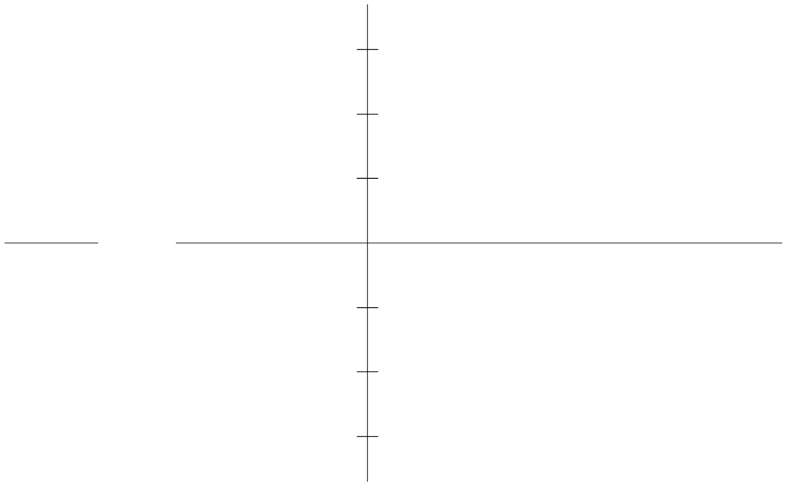
subject to the condition on $0 \le t \le 1$ that $y(t) = f(t)$, a known function. Direct substitution shows that $\exp(W(a)t)$ is a solution of (3). The starting condition can be matched by noting that (1) does not define W uniquely. If complex values are considered, multiple solutions exist, denoted $W_k(a)$, for k an integer. Then, by linearity, a solution of (3) is

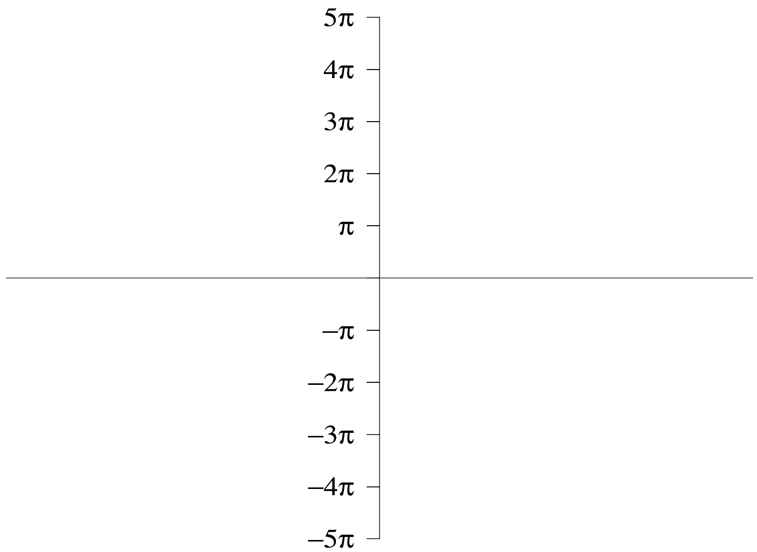
$$y = \sum_{k=-\infty}^{\infty} c_k \exp(W_k(a)t) , \tag{4}$$

and the c_k can be determined to match $f(t)$. One sees immediately that the solution will grow exponentially if any of the $W_k(a)$ have positive real part, and this leads to important stability theorems in the theory of delay equations.

As a prelude to describing the branches W_k, it is useful to consider the branches of some elementary functions that are multivalued. The manipulation of multivalued functions in the complex plane has been the subject of renewed attention in recent years, in an attempt to impose some uniformity on the operations of hand-held calculators and computer programs. As a result, there is now substantial agreement on the principal branches of the elementary functions [7]. The principal argument of a complex number z satisfies $-\pi < \arg z \le \pi$, and the principal branch of its (natural) logarithm is defined to be $\ln z = \ln|z| + i \arg z$. The kth branch of the logarithm is written $\ln_k z = \ln z + 2\pi i k$, implying that $\ln_0 z$ is another representation of the principal branch. (On the matter of notation, notice that although $\log_k$ means logarithm to the base k, there is no standard interpretation of $\ln_k$, so there need be no confusion of meaning.) The notation $\operatorname{Ln} z$ denotes an unspecified branch. The range of each $\ln_k z$ in the complex plane is shown in figure 1. Plotted horizontally is the real part of the logarithm: $\Re \ln_k z = \ln|z|$. The imaginary part, $\Im \ln_k z = \arg z + 2\pi k$, is plotted vertically. The points that form the boundary between two branches belong to the region below them, because of the definition of $\arg z$. This closure also agrees with the 'counter-clockwise continuous' convention (CCC) of Kahan [7].

Figure 2 shows the complex range of each W_k. The horizontal axis is $\xi = \Re W$, the real part of the appropriate branch of W, and the vertical axis is $\eta = \Im W$. The branch boundaries obey either $\eta = 0$ or $\xi = -\eta \cot \eta$ and correspond to cuts in the z plane along portions of the negative real axis. The points on the boundaries belong to the branch below them, which closure rule again satisfies the counter-clockwise continuous (CCC) convention [7]. The negative real axis is divided at $\xi = -1$, with $\xi \ge -1$ belonging to branch 0 and $\xi < -1$ belonging to branch $k = -1$. It should be noticed that the dashed asymptotes at odd multiples of π coincide with the branch boundaries in figure 1. This is a consequence of the fact that for $|z| \to$, $W_k(z) \to \ln_k z$.




5π
4π
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π
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−4π
−5π

Both of these apply for any complex numbers z and w. The derivation of (8) is effected by taking logs of $zw = \exp(\ln z + \ln w)$. The special case $w = 1$ implies

$$\mathcal{K}(\ln z) = 0 \ . \tag{10}$$

While giving an example of working with $\mathcal{K}$, we can derive a result that will be used in section 4. If a and b are complex numbers for which $|\arg b - \arg a| < \pi$, with arg the principal argument, then

$$\ln b = \ln a + \ln \frac{b}{a} \ . \tag{11}$$

The derivation uses (8), (9) and (7).

$$\begin{aligned}\ln b &= \ln a + \ln(b/a) + 2\pi i \mathcal{K}\left(\ln a + \ln(b/a)\right) \\ &= \ln a + \ln(b/a) + 2\pi i \mathcal{K}\left(\ln a + \ln b - \ln a + 2\pi i \mathcal{K}(\ln b - \ln a)\right) \\ &= \ln a + \ln(b/a) + 2\pi i \mathcal{K}(\ln b) - 2\pi i \mathcal{K}(\ln b - \ln a) \ .\end{aligned}$$

The first unwinding number is zero by (10) and the second by the condition on a and b.

3. A new relation for Lambert W

The main result of this article is, in the notation introduced,

$$W_k(z) + \ln W_k(z) = \begin{cases} \ln z \ , & \text{for } k = -1 \text{ and } z \in [-1/e, 0) \ , \\ \ln_k z & \text{otherwise.} \end{cases} \tag{12}$$

The proof takes logarithms of (1). Omitting the argument of $W_k(z)$ for clarity, we have

$$\begin{aligned}\ln z &= \ln\left(W_k \exp W_k\right) = \ln W_k + \ln \exp W_k + 2\pi i \mathcal{K}(\ln W_k + \ln \exp W_k) \\ &= \ln W_k + W_k + 2\pi i \mathcal{K}(W_k) + 2\pi i \mathcal{K}(\ln W_k + W_k + 2\pi i \mathcal{K}(W_k)) \ .\end{aligned}$$

Equation (7) now cancels two of the unwinding numbers.

$$\ln z = \ln W_k + W_k + 2\pi i \mathcal{K}(\ln W_k + W_k) \ .$$

So it is required to show $\mathcal{K}(\ln W_k + W_k) = -k$ for $k \neq -1$, and this requires showing $(2k-1)\pi < \arg W_k + \Im W_k \leq (2k+1)\pi$. Put $z = re^{i\theta}$, with $-\pi < \theta \leq \pi$, and $W_k = \xi + i\eta$ in (1) and separate real and imaginary parts.

$$r \cos\theta = e^{\xi}\left(\xi \cos\eta - \eta \sin\eta\right) \ , \tag{13}$$

$$r \sin\theta = e^{\xi}\left(\eta \cos\eta + \xi \sin\eta\right) \ . \tag{14}$$

For $\theta \neq 0$ or π, divide (13) by (14).

$$\cot\theta = \frac{\cot \arg W_k \cot\eta - 1}{\cot \arg W_k + \cot\eta} = \cot(\arg W_k + \Im W_k) \ .$$

Thus θ and $\arg W_k + \Im W_k$ differ by an integer multiple of π. Furthermore, by continuity, this multiple is the same for all r and θ, except possibly when W is real and negative,

Exponentiating (17) casts it in a more familiar form due to de Bruijn.

$$e^{-v} - 1 + \frac{\ln \ln_k z}{\ln_k z} - \frac{v}{\ln_k z} = 0 \ . \tag{18}$$

From (17), $-\pi \leq \Im v \leq \pi$, and any solution of (18) must satisfy this constraint to be relevant. The last equation has been the starting point for several series expressions, both asymptotic and convergent, for the Lambert W function [6].

References

[1] T.M. Apostol (1974) *Mathematical Analysis*, 2nd Ed. Addison-Wesley, Reading MA.

[2] I.N. Baker and P.J. Rippon (1985) A note on complex iteration. *Amer. Math. Monthly*, **92**, 501–504.

[3] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth (1996) On the Lambert W function. *Adv. Comput. Maths*, in press.

[4] R.M. Corless and D.J. Jeffrey (1992) Well, it isn't quite that simple. *SIGSAM Bulletin*, **26**(3), 2–6.

[5] N.G. de Bruijn (1961) *Asymptotic Methods in Analysis*, North-Holland, Amsterdam.

[6] D.J. Jeffrey, R.M. Corless, D.E.G. Hare & D.E. Knuth (1995) Sur l'inversion de $y^\alpha e^y$ au moyen de nombres de Stirling associés. *C. R. Acad. Sc. Paris, Série I*, **320**, 1449–1452.

[7] W. Kahan (1986) Branch cuts for complex elementary functions. In *The State of the Art in Numerical Analysis:* Proceedings of the Joint IMA/SIAM Conference on the State of the Art in Numerical Analysis, University of Birmingham, April 14-18, 1986, Edited by M. J. D. Powell and A. Iserles, Oxford University Press.

[8] E.M. Wright (1949) The linear difference-differential equation with constant coefficients. *Proc. Roy. Soc. Edinburgh*, A**62**, 387–393.