Some Applications of thethe

the early 1980s, when the program *Maple* defined a function that was named simply W. An historical search, conducted while writing an account of this function [4], found work by the eighteenth century scientist J. H. Lambert that foreshadowed the definition of the function; even though his work did not actually define the function, W was named in his honour. The same search uncovered a fortuitous reason for calling the function W, in that E. M. Wright, a mathematician known for his book with Hardy on pure mathematics, studied the complex values of the function, again without naming it. The function is not connected with the Lambert transform of a function, which has been defined independently [13].

The definition of W is that it is the function that solves the equation

$$We^W = z\ , \tag{1}$$

where z is a complex number. This equation always has an infinite number of solutions, most of them complex, and so W is a multivalued function. The different possible solutions are labelled by an integer variable called the branch of W. Thus the proper way to talk about the solutions of (1) is to say that they are $W_k(z)$, for any $k = 0, \pm 1, \pm 2$, etc. There is always special interest in solutions that are purely real, and so we note immediately that when z is a real number, equation (1) can have either two real solutions, in which case they are $W_0(z)$ and $W_{-1}(z)$, or it can have only one real solution, this being $W_0(z)$ [with $W_{-1}(z)$ now being complex], or no real solution. Even if z is real, the branches other than $k = 0, -1$ are always complex. Admittedly, W does not yet appear on any pocket calculator, but it is known to the computing systems *Maple*, *Macsyma* and *Mathematica* (in the case of Mathematica, the function is called `ProductLog`). Therefore, as soon as a problem is solved in terms of W, numerical values, plots, derivatives and integrals can be easily obtained.

The first physics problem to be solved explicitly in terms of W was one in which the exchange forces between two nuclei within the hydrogen molecular ion (H_2^+) were calculated [11]; this, however, is a long and difficult calculation (and it has already been published) so instead of describing it, we have taken two much simpler problems from standard physics textbooks, problems that many students meet in their physics education, and we have expressed the solutions in terms of W. As mentioned above, the physical content does not change, only the ease of working. An additional point of interest is the fact that the electrostatic application helps to justify a mathematical decision concerning the definition of W that was originally taken entirely on aesthetic (in a mathematical sense) grounds.

2. Wien's displacement law

The spectral distribution of black body radiation is a function of the wavelength λ and absolute temperature T, and is described by $\rho(\lambda, T)$, defined such that $\rho(\lambda, T)\,\mathrm{d}\lambda$ is the power emitted in a wavelength interval $\mathrm{d}\lambda$ per unit area from a black body at absolute temperature T. The wavelength $\lambda_{\max}$ at which ρ is a maximum obeys Wien's displacement law $\lambda_{\max} T = b$, where b is Wien's displacement constant [3]. This law was proposed by Wien in 1893 from general thermodynamic arguments. Once Planck's spectral distribution law is known, Wien's law can be deduced and the value of b determined.

The Planck Spectral distribution law is

$$\rho(\lambda, T) = \quad 8\pi hc/\lambda^5$$

This equation has the trivial solution $x = 0$ and the nontrivial one

$$x = 5 + W_0(-5e^{-5}) \; .$$

Therefore Wien's law is obtained with a new expression for Wien's displacement constant:

$$b = \frac{hc/k}{5 + W_0(-5e^{-5})} = 2.8\;\; 3 \times 10^{-3} \text{ mK}. \tag{3}$$

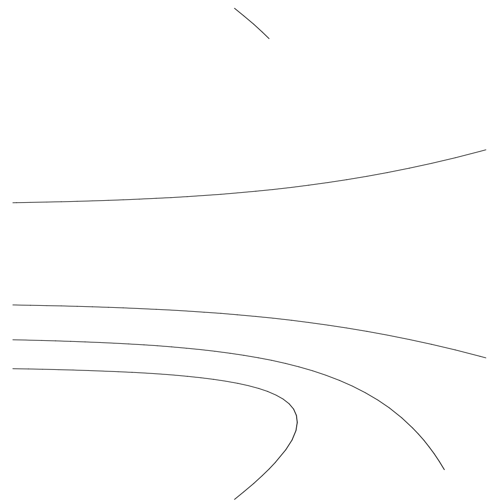
In the past, one would have obtained the numerical value of the law by programming a Newton-Raphson or similar solver on equation (2); now one can start up a computer package and obtain the value without programming. Time is saved not only because no programming is needed, but also because the system developers have implemented the fastest and most accurate method of evaluation.

3. Capacitor fields and conformal mapping

The equipotential lines that are to be calculated are shown in figure 1 in the top set of axes. We see there the fringing field at the edge of a two-dimensional parallel-plate capacitor. The plates are assumed to be semi-infinite, and at potentials $\pm V$. The coordinates of any point in the plane are expressed as a complex number: $\zeta = \xi + i\eta$. The plane is therefore called the ζ-plane, and what is required is a function $\Phi(\zeta)$ giving the electric potential at any point. This function is usually obtained using conformal-mapping techniques [12]. In general, conformal techniques solve a problem by relating its geometry to a simpler geometry in which the governing

Physical plane:

(The ζ-plane)



$$
\begin{aligned}
e^{\zeta-1} &= e^{z}e^{e^{z}} , \\
e^{z} &= W_k(e^{\zeta-1}) , \\
\zeta - 1 - z &= W_k(e^{\zeta-1}) , \\
z &= \zeta - 1 - W_k(e^{\zeta-1}) .
\end{aligned}
\tag{5}
$$

Some further analysis (not given here) shows that the restriction $-\pi < \Im z < \pi$ implies that the branch index k is also specified, once ζ is known; moreover, we can give an analytic formula for this, in terms of $\mathcal{K}$, the *unwinding number* [5, 7]. The expression is

$$
k = \mathcal{K}(\zeta) = \left\lceil \frac{\Im\zeta - \pi}{2\pi} \right\rceil . \tag{6}
$$

Here the symbol $\lceil\ \rceil$ denotes the ceiling function, which is the integer obtained by rounding *up* (as opposed to floor which is obtained by rounding down).

Figure 2 shows how the inverse transformation works. Recalling the notation, $\zeta = \xi + i\eta$, we divide the ζ-plane into strips of width 2π. The main strip between the plates, and extending to the right, is $-\pi < \eta < \pi$ and is shown containing solid lines. The strips $-3\pi < \eta < -\pi$ and $\pi < \eta < 3\pi$ are shown containing dashed lines. Each strip is transformed using a different branch of W, the one with index $k = \mathcal{K}(\zeta)$, onto a distinct portion of the strip $-\pi < \Im z < \pi$. The portions of the strip thus mapped are symmetric, in the sense that W_{-k} and W_k map into regions symmetric about the real z axis.

In summary, we have derived the following new analytical formula for the solution for the fringing fields of a semi-infinite capacitor. The potential at the point ζ is

$$
\Phi = (V/\pi)\Im\left[\zeta - 1 - W_{\mathcal{K}(\zeta)}\left(e^{\zeta-1}\right)\right] . \tag{7}
$$

As stated in the introduction, for this formula to be actually useful, it must be easily evaluated. Although the number of computer packages that contain W built-in is still small, the packages are among the most popular ones at the moment. Therefore this formula is genuinely computational.

This application to conformal mappings adds an interesting postscript to the history of the definition of W. The equation (1) does not by itself completely define the branches of W [4, 6], as explained in the next it must Td(v)Tj 0ation3M50 Td(6n)1.0799 0 Td(the)Tj 98399 0 Td(Tm(.)T 41d(xri7h/R11 9d)Tj 37.3199 Td(

Physical plane:

Lines in ζ-plane
where values of

x	$W_0(x)$	$W_{-1}(x)$	$W_1(x)$
e	1	$-0.5321 - 4.597\,i$	$-0.5321 + 4.597\,i$
1	0.5671	$-1.534 - 4.375\,i$	$-1.534 + 4.375\,i$
0	0	complex infinity	complex infinity
$-1/e$	-1	-1	$-3.089 + 7.462\,i$
$-1/4$	-0.3574	-2.153	$-3.490 + 7.414\,i$
$-1/4 + i$	$0.3169 + 0.6807\,i$	$-0.9667 - 2.532\,i$	$-1.843 + 6.241\,i$
$-1/4 - i$	$0.3169 - 0.6807\,i$	$-1.843 - 6.241\,i$	$-0.9667 + 2.532\,i$

Table 1. Some exact and approximate values for the Lambert W function. Of the infinite number of branches W_k, we tabulate 3 branches. The entries 'complex infinity' mean that the values of $W_{-1}(0)$ and $W_1(0)$ have infinite real part, but their imaginary parts depend upon the direction in which 0 is approached.

The Lambert W function has a rich variety of applications