Editor's Corner: The Unwinding Number

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1 Introduction

From the Oxford English Dictionary we find that *to unwind* can mean "to become free from a convoluted state". Further down we find the quotation "The solution of all knots, and unwinding of all intricacies", from H. Brooke (The Fool of Quality, 1809). While we do not promise that the unwinding number, defined below, will solve *all* intricacies, we do show that it may help for quite a few problems.

The purpose of this column is to see exactly how these identities have to be modified, once we choose the principal branch of the logarithm. Introducing the unwinding number $\ (z)$ turns out to be sufficient for this purpose.

1.2 Unwinding number

We define the *unwinding number* $\ (z)$ by

$$\ln(e^z) = z + 2\pi i \ \ (z) \, . \tag{5}$$

See [4], where this function is used to derive new identities for the Lambert W function.

Functions similar to   have been defined several times in the literature. In 1974, Apostol [1] briefly considered a cognate of . Charles Patton has defined several functions including $\mathrm{UNLN}(z) = \ln \exp z - z$ (see [8] for a brief discussion of UNLN) which is $2\pi i \ \ (z)$ in our notation. Aslaksen [2] defines several functions including $\mathrm{Imq}(z)$, which turns out to be $- \ \ (z)$ in our notation. It would be interesting to see the results of a thorough historical investigation.

One can define $\ (z)$ without logarithms by using the floor function. If $\Im(z)$ is the imaginary part of z, then

$$\ (z) = \ \ (i\Im(z)) = \left\lfloor \frac{\pi - \Im(z)}{2\pi} \right\rfloor \, . \tag{6}$$

It is easy to see that $\ (z) = 0$ if $-\pi < \Im(z) \leq \pi$, and in general that $\ (z) = -n$ if $(2n-1)\pi < \Im(z) \leq (2n+1)\pi$. Thus the unwinding number is constant on horizontal strips. Note the closure on the top of the strips.

The function was called the 'unwinding number' because we thought of $\exp z$ as winding z around the branch point of log; in order to get z back one has to 'unwind'.

2 Connection with the Riemann surf ma

5. **Theorem** These all follow on writing a^b as its definition $\exp(b\ln(a))$:

   (a) $\ln(z^w) = w\ln z + 2\pi i \quad (w\ln z)$

   (b) $(z_1 z_2)^w = z_1^w z_2^w \exp(2\pi i w \quad (\ln z_1 + \ln z_2))$ (the generalization to n terms in the product is immediate)

   (c) $(z^v)^w = z^{vw}\exp(2\pi i w \quad (v\ln z))$ (notice that the order is important, and we ascribe our conventional meaning to z^{vw}).

5 Applications

In this section we give some sample applications, to show that this is not just an empty definition.

5.1 Fateman's z^w problem

Consider

$$y = z^w \tag{11}$$

as an equation for z, given y and w in $\mathbb{C}$, as discussed in [5]. We divide this into two problems: we first try to decide when $\zeta = y^{1/w}$ solves equation (11). This will give sufficient conditions for the classical formula to be true. We then try to discover all roots of (11), which turns out to be harder.

5.1.1 Sufficient conditions

Let $\zeta = y^{1/w}$. Then $\zeta^w = \exp(w\ln\zeta)$ or

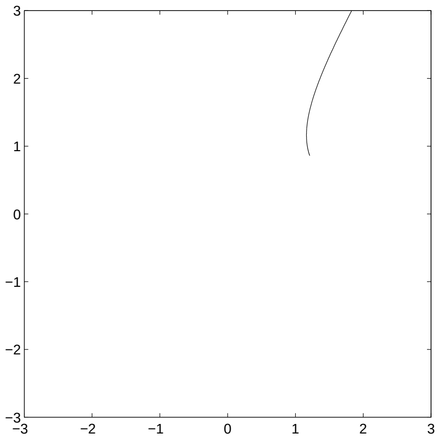
$$\begin{aligned}
\zeta^w &= \exp(w\ln\exp(\frac{1}{w}\ln y)) \\
&= \exp(w(\frac{1}{w}\ln y + 2\pi i \quad (\frac{1}{w}\ln y)) \\
&= y\exp(2\pi i w \quad (\frac{1}{w}\ln y))\ .
\end{aligned}$$

This is equal to y if and only if $w \quad ((\ln y)/w)$ is an integer, say n. If w (which is given for the problem) is irrational, then n and hence must be zero. If w is rational, then one can show by pigeonhole arguments that must still be zero.

So ζ is a root of $z^w = y$ if and only if $((\ln y)/w) = 0$, or y is in the clearcut region for $(\ln y)/w$. This can happen if and only if $(\ln y)/w = t + ip$ where $-\pi < p \le \pi$. This implies that $\ln y = w(t+ip) = (a+ib)(t+ip) = (at-bp) + i(bt+ap)$ or, with $\ln y = \ln s + i\theta$ giving the polar coordinates of y, $s = \exp(at-bp)$ and $\theta = bt + ap$. If $b \ne 0$ we can eliminate the parameter t to get

$$s = e^{a\theta/b-(a^2+b^2)p/b}\ .$$

Remembering that $-\pi < p \le \pi$, then, if $ab \ne 0$, this is a domain bounded by logarithmic spirals. For Fateman's s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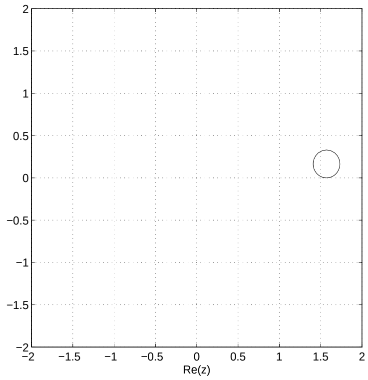
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Re(z)

2. that it gives a precise definition for 'simplify/symbolic' in Maple or other CAS, that can be used for provisos: one rewrites an expression using , one uses whatever assumptions one has to evaluate as many instances of as possible, and than one sets to zero whatever unwinding numbers are left. The proviso for the result is then just the unwinding numbers that had been set to zero.

The principal *disadvantages* in using in a computer algebra system include

1. that rewrite rules using essentially double the size of the printed output (though not the DAG), giving answers of the form $Y + 2\pi i \ (Y)$, and

2. that the rules for removal of are essentially geometric and need decisions to be taken on the basis of where its arguments are in $\mathbb{C}$.

Automatic geometric reasoning with elementary functions is not well understood yet, and indeed this may prove to be a "grand challenge" to symbolic computation systems, with many other possible applications. Perhaps we may turn this disadvantage of into a stimulant for development in this area.

More work needs to be done before this function can be properly implemented. We invite discussion of this function, and in particular we invite discussions containing trial implementations in real computer algebra systems. The primary purpose of this present article is to help to get people *used to* the idea of the unwinding number; of course such a psychological adjustment—to learn to think of as an answer, not a question—is a necessary preliminary to its being used in practice. We invite you to check the results in this paper, and to draw some clearcut regions for yourselves (e.g. for $\sqrt{1-z^2}$ or the hyperbolic functions) to help make that adjustment.

Mathematicians make progress by turning analysis into algebra. We hope that (z) will help to turn complex analysis into computer algebra.

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References

[1] T. M. Apostol, Mathematical Analysis, 2nd ed., Addison-Wesley, 1974.

[2] H. Aslaksen, "Multiple-valued complex functions and computer algebra", this BULLETIN, *pp.* 12–20.

[3] Robert M. Corless, Gaston H. Gonnet, D. E. G. Hare, D. J. Jeffrey, & D. E. Knuth, "On the Lambert W Function", Advances in Computational Mathematics, *in press*.

[4] D. J. J