

2. Real solutions and rational solutions. In (1), the case $b < 1$ can be avoided by the transformation $x \rightarrow -x$, so only the case $b > 1$ is considered.

Theorem 1. Let $b, c \in \mathbb{R}$, $b > 1$ and $n \in \mathbb{Z}$. Let $\sqrt[n]{c}$ denote the real branch of $c^{1/n}$. The solutions $x \in \mathbb{R}$ of the equation $x^n b^x = c$ are as follows.

1. If n is odd and $c > -(n/e \ln b)^n$ or if n is even and $c \geq 0$,

$$x = \frac{n}{\ln b} W_0 \left(\frac{\ln b}{n} \sqrt[n]{c} \right) .$$

2. If n is odd and $0 > c > -(n/e \ln b)^n$,

$$x = \frac{n}{\ln b} W_{-1} \left(\frac{\ln b}{n} \sqrt[n]{c} \right) .$$

3. If n is even and $0 < c \leq (n/e \ln b)^n$,

$$x = \frac{n}{\ln b} W_0 \left(-\frac{\ln b}{n} \sqrt[n]{c} \right) , \quad \text{or} \quad x = \frac{n}{\ln b} W_{-1} \left(-\frac{\ln b}{n} \sqrt[n]{c} \right) .$$

Proof: Consider the case for n odd. Take the real-branch n th root of both sides of (1). Then it becomes, because n is odd,

$$x b^{x/n} = x e^{(x/n) \ln b} = \sqrt[n]{c} ,$$

and multiplying through by $(\ln b)/n$ and comparing with (2) establishes that

$$x = \frac{n}{\ln b} W_k \left(\frac{\ln b}{n} \sqrt[n]{c} \right) .$$

Only the branches $k = 0$ and $k = -1$ of W take real values [1]. The principal branch $W_0(x)$ is real for $x \geq -1/e$, while $W_{-1}(x)$ is real for $-1/e < x < 0$. Imposing these restrictions leads to the cases listed in the theorem. The other cases are similar. It is clear that the converse statements are also true.

Example. The solutions of $x^2(9/16)^x = 3/16$ are

$$\begin{aligned} \frac{-2}{\ln(16/9)} W_0 \left(\frac{\sqrt{3}}{8} \ln \frac{16}{9} \right) &\approx -.387351650 , \\ \frac{-2}{\ln(16/9)} W_0 \left(-\frac{\sqrt{3}}{8} \ln \frac{16}{9} \right) &= \frac{1}{2} , \\ \frac{-2}{\ln(16/9)} W_{-1} \left(-\frac{\sqrt{3}}{8} \ln \frac{16}{9} \right) &\approx 11.35508246 . \end{aligned}$$

One must be aware of the possibility of equivalent expressions:

$$\frac{-2}{\ln(16/9)} W_0 \left(-\frac{\sqrt{3}}{8} \ln \frac{16}{9} \right) = \frac{-1}{\ln(4/3)} W_0 \left(-\frac{\sqrt{3}}{4} \ln \frac{4}{3} \right) = \frac{1}{2} .$$

We are led by the above to seek an algorithm for deciding whether $W_k(r_1)$

Proof: By theorem 1, x as defined in (4) is a solution of

$$(6) \quad x^n (\text{ipf } b)^x = c = r_1^n n^n (\text{ipc } b)^n r_2 .$$

Then $x \in \mathbb{Q} \Rightarrow x \in \mathbb{Z}$, because if $x = p/q$, with $p, q \in \mathbb{Z}$, then

$$(\text{ipf } b^p)^{1/q} = c(q/p)^n .$$

Clearly the right-hand side is rational, but by the construction of $\text{ipf } b$, the left side is not in \mathbb{Q} unless $q = 1$.

Computational algorithm. A practical procedure is as follows. The quantity x defined in (4) is evaluated in two stages. At first, the order of magnitude of x is determined; let it be $O(10^p)$. The evaluation is then repeated using more digits of precision than p , so that if x is indeed an integer, its value will be determined correctly.¹ The proposed integral value is then accepted if it satisfies (5), otherwise there is no rational simplification. If a computer system verifies (5) by direct computation of the left-hand side, very large integers are generated. It would therefore be wise for a system to use the laws of exponents first.

Example. Returning to problem (3), one now computes

$$\frac{10561}{\ln(3/2)} W_0 \left(\frac{528309}{168976} \ln \left(\frac{9}{4} \right) 3^{\frac{7451}{10561}} 2^{\frac{3110}{10561}} \right) \approx 39135.$$

Here 5 digits of precision have been used. The size of the result shows that a computation accurate to 6 digits is sufficient (less than half the 16 needed by the continued fraction method), and this gives 39134.0 as the result. We now must verify that

$$(3/2)^{39134} - \left(\frac{528309}{168976} (10561) 2 \right)$$

and the symmetric solution (y, x) . This implies simplification rules for W . We find from the $y = x$ solution, setting $y = 1/t$,

$$\begin{aligned} W_0(t \ln t) &= \ln t, & \text{for } t > 1/e, \\ W_{-1}(t \ln t) &= \ln t, & \text{for } 0 < t < 1/e. \end{aligned}$$

From the parametric solution, we obtain the additional relations

$$\begin{aligned} W_0\left(-\frac{\ln(1+s)}{s(1+s)^{1/s}}\right) &= -\frac{1+s}{s} \ln(1+s) & \text{for } -1 < s < 0, \\ W_{-1}\left(-\frac{\ln(1+s)}{s(1+s)^{1/s}}\right) &= -\frac{1+s}{s} \ln(1+s) & \text{for } 0 < s. \end{aligned}$$

These results are equivalent to a result by Lauerier [3, 4]. Equations (54–57) in [3] are obtained by writing $s = e^{2\Delta} - 1$. As of this writing, there are no computer algebra implementations of these (domain-specific) simplification rules.

4. The complex case. If $\text{cas } e$. If $\text{cas } e$. \mathfrak{R}

Example. For the equation $x^{1/3}256^x = 1$, the exponential factor in (8) is 1, but even so, W_0 and W_3 give solutions, but W_{-1} , W_1 , W_2 do not, unless $x^{1/3}$ takes values from its other branches.

References

- [1] CORLESS, R. M., GONNET, G. H., HARE, D. E. G., JEFFREY, D. J., AND KNUTH, D. E. On the Lambert W function. *Advances in O**f** E**f** E* *Vin O**f** E* **C**^Y