Advances in Computational Mathematics, 5 (1996) 329 359.

On the Lambert W Function

. M. C [1], G. H. G [2], D. E. G. H [3], D. J. J [1] D. E. K [4]

[1] *Department of Applied Mathematics, Universit of Western Ontario, London, Canada, N6A 5B*

[2] *Institut fur Wissenschaftliches Rechnen, ETH Zurich, Swit erland*

[3] *S mbolic Computation Group, Universit of Waterloo, Waterloo, Canada, N2L 3G1*

[4] *Department of Computer Science, Stanford Universit , Stanford, California, USA 94305-2140*

Abstract

L W fi w 7! we^w. I , fl . W, ffi , W.

1. Introduction

I 1758, L $x = q + x^m$ - x q. L ; x [48,49]. I [28], E L '

$$x \quad x = (\quad)vx \quad {}^{+} \tag{1:1}$$

L , x^{-} x $m =$ $q = (\quad)v$. E L

$$\begin{aligned} x^n = 1 + nv &+ \tfrac{1}{2}n(n + \quad + \quad)v^2 \\ &+ \tfrac{1}{6}n(n + \quad + 2 \quad)(n + 2 \quad + \quad)v^3 \\ &+ \tfrac{1}{24}n(n + \quad + 3 \quad)(n + 2 \quad + 2 \quad)(n + 3 \quad + \quad)v^4 \\ &+ \quad : \end{aligned} \tag{1:2}$$

A , E , = . (1.1) ( ) !

$$x = vx \quad : \tag{1:3}$$

E (1.3) $= 1$,

6= 0. , (1.3) , x x , $z = x$
$u =$ v. $z = uz$, (1.3) = 1.
(1.2), E fi = = 1 (1.2)
$(x^n$ 1)=n. N n = 0 x -
- :

$$x = v + \frac{2^1}{2!}v^2 + \frac{3^2}{3!}v^3 + \frac{4^3}{4!}v^4 + \frac{5^4}{5!}v^5 + \quad : \qquad (1{:}4)$$

, jvj < 1=e, fi $T(v)$
tree function [41]. I $W($ $v)$, $W(z)$ fi

$$W(z)e^{W(z)} = z \; : \qquad (1{:}5)$$

W T, W.
: ,
[10, 13, 25, 41, 67]; [64];
R´ [56, P III.209, . 146].
W, , -
ffi [69]. I [30], F , C
fi - $W(x)$ x > 0.
M -
W , 2
[14].

, , ,
, W . -
W , B [11, . 27 28],
fi 0. ffi
(M 2) -
fl - $W(z)$. F , K. B.
C F M ([58])
,
W.

notation

W , M .
Lambert W function, v = 17 9 963 03 89 00 [F2 93(2)F9 963 1019 7[(544)] 33 2(F6 9

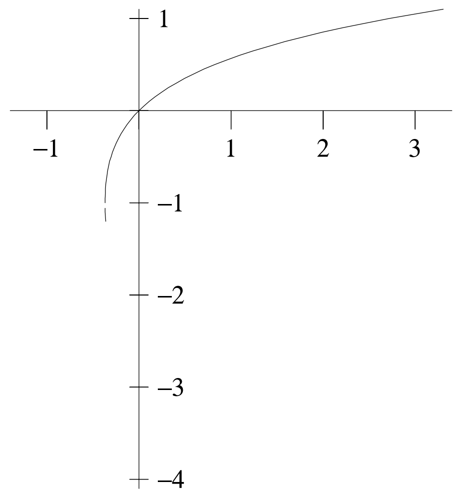


Figure 1. $W(x)$. , $W_0(x)$; , $W_{-1}(x)$.

, $W(x)$ 1 $W_{-1}(x)$. $W_0(x)$ *principal branch* W . 4.

2. Applications

W , P ' ( ) W . I , W . .

Combinatorial applications

. L t_n n . $T(x) =$ $t_n x^n = n!$ fi $T(x) = x + xT(x) + xT(x)^2 = 2! +$ $=$ $xe^{T(x)}$, $T(x)e^{-T(x)} = x$ $T(x) =$ $W($ $x)$. I [57], R L $t_n = n^{n-1}$, [10,13,67].

I

$$U(x) = T(x) \quad \tfrac{1}{2}T(x)^2 ; \qquad (2{:}1)$$

$U(x)$ *unrooted* [24,41]. ,

$$V(x) = \tfrac{1}{2} \quad \frac{1}{1 \quad T(x)} \qquad (2{:}2)$$

; $\tfrac{1}{2}T(x) + \tfrac{1}{4}T(x)^2$

Solution of a jet fuel problem

E_t R [3, pp. 312-323]. fi fi c_t w_0 w_1,

$$E_t = \frac{C_L}{c_t C_D} \quad \frac{w_0}{w_1} \quad ; \tag{2.6}$$

$$R = \frac{2}{c_t C_D} \quad \frac{2C_L}{S}^{1=2} \quad w_0^{1=2} \quad w_1^{1=2} \quad ; \tag{2.7}$$

E_t, C_L C_D ffi, w_0, w_1 fi, R, S . fi . $w = w_1 = w_0$ $c = E_t C_D c_t = C_L$

$$A = \frac{p\overline{2}E_t}{R} \quad \frac{w_0}{SC_L}^{1=2} : \tag{2.8}$$

$c =$ w

$$2A \frac{1 \quad p\overline{w}}{w} = 1 : \tag{2.9}$$

$A < 0$, w. I W,

$$w = \begin{cases} A^{-2} W_0^2 \left(Ae^A\right) \; ; & A \quad 1, \\ A^{-2} W_{-1}^2 \left(Ae^A\right) \; ; & 1 \quad A < 0. \end{cases} \tag{2.10}$$

O w , fi $c =$ w.

Solution of a model combustion problem

$$\frac{dy}{dt} = y^2(1 \quad y); \quad y(0) = " > 0 \tag{2.11}$$

[54,59] . , W, [54] . $y(t)$ ([54]):

$$\frac{1}{y} + \quad \frac{1}{y} \quad 1 = \frac{1}{"} + \quad \frac{1}{"} \quad 1 \quad t \; ; \tag{2.12}$$

$$s = W_k(a) ;$$

W_k. (4 W.)
$(W_k(a)t)$ $y = ay(t \quad 1)$,

$$y = \sum_{k=-\infty}^{\infty} c_k \quad (W_k(a)t) ; \qquad (2{:}18)$$

c_k (. .). O
$W_k(a)$,
[8].
ffi ,
,

, fi R.

Similarity solution for the Richards equation

W [7]. B , *moisture tension* Ψ,

$$\frac{d}{d\Psi}\frac{@\Psi}{@t} = \frac{@}{@z}\left[K(\Psi)\frac{@\Psi}{@z} \quad K(\Psi)\right] ; \qquad (2{:}24)$$

, , ff

$$^{*}A\frac{dA}{dt} = 1 \quad A \qquad (2{:}25)$$

W

$$A(t) = 1 + W \left(\quad (\quad 1 + A(0)) \qquad \frac{(A(0) \quad 1) \ ^{*} \quad t}{1 \quad A(0)} \right] : \qquad (2{:}26)$$

B W . W_0 *capillar rise*, W_{-1} *infiltration*.

Volterra equations for population growth

I [22, 102 109], fi

$$\frac{dx}{dt} = ax(1 \quad y); \quad \frac{dy}{dt} = \quad cy(1 \quad x) ; \qquad (2{:}27)$$

W ((11) (12) 104 [22]). , $a = 2$ $c = 1$, B1 DE E ff [40]. . y^{+} y^{-},

$$y^{+} = \quad W_{-1} \quad Cx^{-c=a}e^{cx=a} \quad ; \qquad (2{:}28)$$
$$y^{-} = \quad W_0 \quad Cx^{-c=a}e^{cx=a} \quad ;$$

C .

fi t, fi :

$$t = \int_{x_0}^{x} \frac{d}{a \ (1 \quad y(\ ))} = \int_{y_0}^{y} \frac{d}{c \ (1 \quad x(\ ))} : \qquad (2{:}29)$$

, -
.

$c_0; c_1; : : :$

, . I ,

.

I L [12].

W fi .

3. Calculus

W 0. L (. [12]), $W_0(z)$:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(\ \ n)^{n-1}}{n!} z^n : \qquad (3{:}1)$$

I we^w.

B W_0 $z =$ $1{=}e$ ($w \mapsto we^w$ 0 $w =$ 1), (3.1) $1{=}e$. $1{=}e$.

, E (1.2)

$$M_E(\ ;\ )|v| < \frac{1}{e};$$

$M_E(a;b) = \sqrt{(a^2+b^2){=}2}$ E a b. , . F v, = , . , W (1.2) = = 1 W . D ff fi $x = W(x)e^{W(x)}$ W W', W:

$$\begin{aligned} W'(x) &= \frac{1}{(1+W(x))\ \ \ (W(x))} \\ &= \frac{W(x)}{x(1+W(x))}; \quad x \neq 0. \end{aligned} \qquad (3{:}2)$$

H , . , M 3 $W(x){=}(x(1+W(x)))$ ff W, $W_0'(0) = 1$. [20]

[illegible] ([illegible]) [illegible]

[illegible] , [illegible] n [illegible] W

$$\frac{d^n W(x)}{dx^n} = \frac{(\quad nW(x))p_n(W(x))}{(1+W(x))^{2n-1}} \quad [illegible]\ n \quad 1\ ; \tag{3:3}$$

[illegible] $p_n(w)$ [illegible]

$$p_{n+1}(w) = \quad (nw + 3n \quad 1)p_n(w) + (1+w)p_n'(w)\ ; \quad [illegible]\ n \quad 1\ : \tag{3:4}$$

[illegible] $p_1(w) = 1$. [illegible] $p_n(0)$ $(\quad n)^{n-1}=n!$ [illegible] n 1. A [illegible] p_n [illegible] , [illegible]

$$\frac{d^n W(e^x)}{dx^n} = \frac{q_n\left(W(e^x)\right)}{\left(1+W(e^x)\right)^{2n-1}} \quad [illegible]\ n \quad 1\ ; \tag{3:5}$$

[illegible] q_n, [illegible]

$$q_n(w) = \sum_{k=0}^{n-1} \left\langle \begin{matrix} n & 1 \\ & k \end{matrix} \right\rangle (\quad 1)^k w^{k+1}\ ; \tag{3:6}$$

[illegible] E[illegible] [35]. [illegible]

$$q_{n+1}(w) = \quad (2n \quad 1)wq_n(w) + (w+w^2)q_n'(w) \tag{3:7}$$

[illegible] $q_1(w) = w$.

[illegible] $We^W = z$ [illegible] , [illegible]

$$[illegible]\ W(z) = [illegible]\ z \quad W(z)\ ; \tag{3:8}$$

[illegible] $z > 0$. [43] [illegible]

[illegible] W [illegible].

I [58], K. B. [illegible] N[illegible] :

$$x = pe^p\ ; \tag{3:9}$$

$$\frac{dy}{dx} = p\ : \tag{3:10}$$

D[illegible] (3.9) [illegible] y [illegible]

$$\frac{dx}{dy} = \frac{dp}{dy}e^p + pe^p\frac{dp}{dy}\ ; \tag{3:11}$$

(3.10)

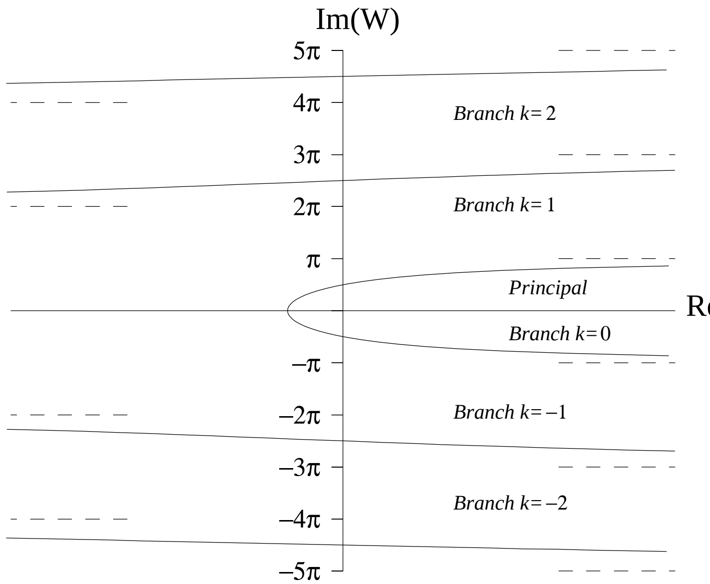
$$\frac{dy}{dp} = p(p+1)e^{p} \; ; \qquad (3:12)$$

$$y = (p^2 \quad p+1)e^{p} + C \, : \qquad (3:13)$$

y $W(x)$, $W(x)$. I ,

$$W(x)\,dx = (W^2(x) \quad W(x)+1)e^{W(x)} +$$

Figure 3. … $p =$ … $z.$ … $z = e^p$ … F… 2. … C … D …


Im(W)
5π
4π
3π
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π
−π
−2π
−3π
−4π
−5π
Branch k = 2
Branch k = 1
Principal
Branch k = 0
Branch k = −1
Branch k = −2

Figure 4. [illegible] $W(z)$. [illegible] 0. [illegible]

[illegible]

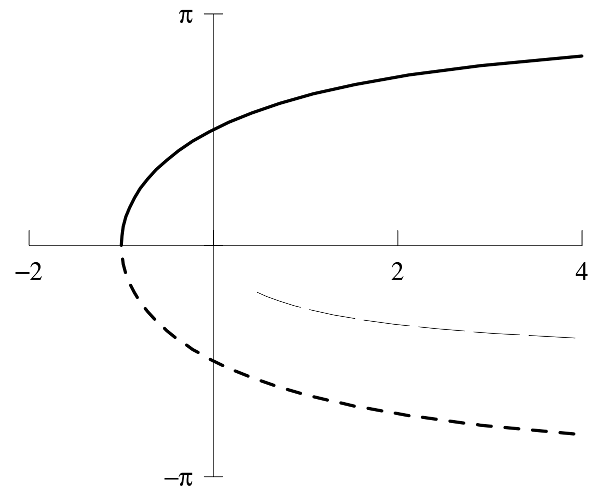

Figure 5. , $W_0(z)$. A, B, C, D Figure 6. Figure 6.

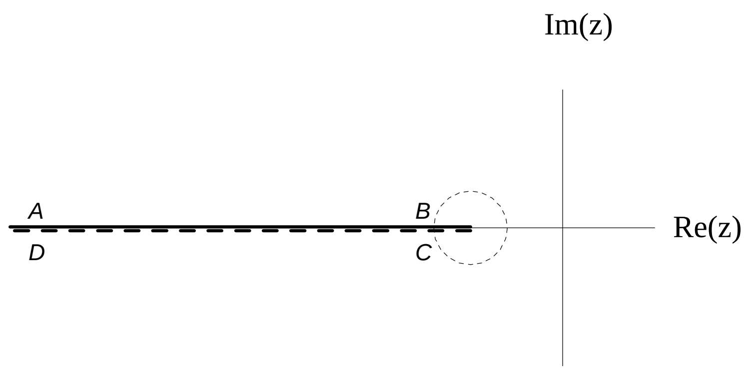

Figure 6. $W_0(z)$. $z = we^w$ Figure 5, . C B Figure 5, B. $\{z : -\infty < z \le -1/e\}$.

, $W_0(z)$ (CCC) $z = -1/e$. $z = -1/e + \epsilon e^{it}$ z- , $-\pi < t$, 0. Figure 5 Figure 6.

B , W_{-1} W_1 , $\{z : -\infty < z \le -1/e\}$ $\{z : -1/e < z \le 0\}$. . CCC $z = 0$. $W_{-1}(z)$ $z \in [-1/e; 0)$. $W_0(z)$ $W_{-1}(z)$ W . Figure 7; Figure 8, Figure 9. A W $\{z : -\infty < z \le 0\}$, , .

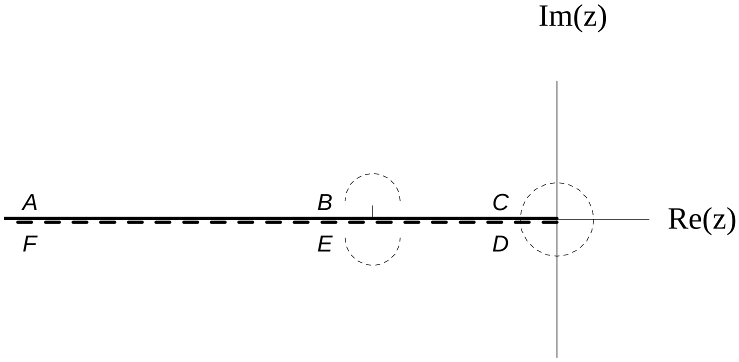

Figure 7. $W_k(z)$, $k \neq 0$. C F $W_1(z)$ $W_{-1}(z)$, $z = -1/e$ $z = we^w$ F 8 9. D C, C, D C F 8 9.

Figure 8. D $W_{-1}(z)$. fi , A, $W = -1$ $z = (-\infty; -1/e]$. $W = -1$, C fi $z = [-1/e; 0)$. , $W_{-1}(z)$. CD .

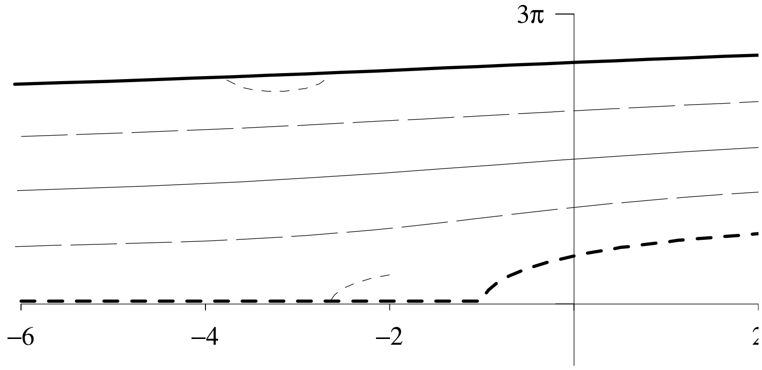

Figure 9. D $W_1(z)$. A C z. N $W_1(z)$,

(4.22), $\{z: -1 < z \le -1/e\}$ $\{z: 0 \le z < \infty\}$.

Asymptotic expansions

W 0 ( ) fi , $w \mapsto we^w$. $z = 0$.)

$$w = W(z) = \ln z + u\,; \tag{4.6}$$

z 0 u z, fi W, $we^w = z$,

$$(\ln z + u)ze^u = z\,; \tag{4.7}$$

$$(1 + \frac{u}{\ln z})e^u = \frac{1}{\ln z}\,. \tag{4.8}$$

$|u| \ll |\ln z|$ $e^u \sim 1/\ln z$,

$$u \sim -\ln \frac{1}{\ln z}\,; \tag{4.9}$$

(4.9) . H , u fi (4.9), ,

[illegible] (4.9) [illegible] :

$$u \quad [illegible] \quad \frac{1}{[illegible]\ z} : \tag{4:10}$$

O [illegible] z [illegible] $\frac{1}{\mathrm{Log}\, z}$ [illegible] z. [illegible] z [illegible] z [illegible] 1. K [illegible] B [illegible] [11, [illegible] . 27 28] [illegible] W [illegible] 0 [illegible] fi [illegible] . (B [illegible] $z \to \infty$, [illegible] $z \to 0$ [illegible] . B [illegible] L [illegible] I [illegible] .) I [illegible] .

C [illegible] (4.6) [illegible] (4.10), [illegible]

$$w = [illegible]\ z \quad [illegible]\ z + v : \tag{4:11}$$

O [illegible] (4.11) [illegible] $we^w = z$ [illegible]

$$\frac{([illegible]\ z \quad [illegible]\ z + v)e^v z}{[illegible]\ z} = z : \tag{4:12}$$

F [illegible] 1= [illegible] z [illegible] z= [illegible] z [illegible], [illegible] B [illegible] . [illegible], [illegible] $z \neq 0$ [illegible] $z \neq 0$,

$$e^{-v} \quad 1 \quad v + \quad = 0 : \tag{4:13}$$

[illegible] (2.4.6) B [illegible], [illegible]

Remark [illegible]

(4:

(4.18), $z + 2 \pi i k$ k,

$$W_k(z) = \ln z + 2\pi i k - \ln(\ln z + 2\pi i k) + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \ln^m(\ln z + 2\pi i k)(\ln z + 2\pi i k)^{-k-m} : \qquad (4{:}20)$$

k; ... $z \to 1$, ... A ... $W_{-1}(x)$ ... $x < 0$. ... $(-x)$ ... (z) ... $(-x))$ ... $(z))$. A ... (4.13) ... v, ... A ... , ... x ... W:

Series expansions about the branch point

$p = +\sqrt{2(ez+1)}$ $We^W = z$, ... $1 + W$,

$$\frac{p^2}{2} - 1 = We^{1+W} = -1 + \sum_{k \geq 1} \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right) (1+W)^k ; \qquad (4{:}21)$$

$$W(z) = \sum_{\ell=0}^{\infty} \mu_\ell p^\ell = -1 + p - \tfrac{1}{3} \ldots$$

F W_{-1} $p = \sqrt{2(ez+1)}$, I (z) 0. F I (z) < 0, $W_1(z)$. $1/e$

W_{-1} W_0 [45] K ffi c_n

$$= \; + \tfrac{2}{3} \;^2 + \tfrac{4}{9} \;^3 + \tfrac{44}{135} \;^4 + \; = \sum_{n\geq 1} c_n \;^n ; \qquad (4.26)$$

$$(1+ \;)e^{-} = (1 \;)e \; ; \qquad = \; + O(\;^2) : \qquad (4.27)$$

[41, 323]. H c_n n 15

$$\frac{1}{n} \quad c_n \quad \frac{1}{n} + \frac{1 \quad 1/n}{n(1+1/2+ \quad +1/n)} : \qquad (4.28)$$

I M 3, W . P M 2, 7! e [1; 1) , . I M 2, (1; 1/e] (0; 1), fi - fi W M 3.

5. Numerical Analysis

I [27], E $x = a^x$ a, fi $W_k(x)$ x L [51,52]. [36] H fi $W_k(x)$ x , . , [70], [71] .

W : fi , *arbitrar* , M . fi W, ffi . (. [16, . 14]) $C = xf'$

ff z.

W $z = 1/e$ W_0, W_{-1}

W_1), $W(z)$ 1. I ,

$z = 0$; *relative*

$W($

$jW'(z)j =$. O [illegible] $W_k(z)$ [illegible] $W_{-k}(z)$, [illegible] $\overline{W_k(z)} = W_{-k}(\overline{z})$ ([illegible] 4). F [illegible] $W_k(z)$ [illegible] k

I ... fi ...
fi ... : ... n ... ,
... k ... n ... $(k\ \ 1)$,
... d ... ,
... d ... ,
... $d{=}n$ ... , ... $d{=}n^2$ ... , ... F ... ,
... $(k\ \ 1)$ ... d_{k-1} ... , ... k ... ,
... $(n\ \ 1)d_{k-1}$ ... , ... fi
d_{k-1} ... ff ... ,
... nd_{k-1} ... . [66], ...
N ... -
... .
... , ... 1 ... -
... . F ... n ...

. I ... M ... $W_0(x)$, $-1/e \le x$, ... $W_{-1}(x)$, $-1/e \le x < 0$, ... A ... W ... F ... W ... , H ...

$$w_{j+1} = w_j - \frac{w_j e^{w_j} - z}{e^{w_j}(w_j+1) - \dfrac{(w_j+2)(w_j e^{w_j} - z)}{2w_j+2}} . \tag{5.9}$$

I ... M ... , ... F ... z, ... ffi ... fi ... (4.18). ... $W_0(z)$, $W_{-1}(z)$ ... $W_1(z)$ ... z ... $-1/e$, fi ... (4.22) ... W_{-1} ... W_1). ... $W_0(z)$ ... 0, ... P ... (M ... (3,2)-P ...). F ... , ... $W_{-1}(z)$ ... z ... 0 ... $-1/e$ ... W_{-1} ... , ... z ... $-1/e$ ... . A ... W_1. F ... M ... , ...

6. Concluding Remarks

... L ... W ... , ... W ... ; ... , ... : A ... W ... (... , ... $T(x)$). N ... . L ... W ... fi ... , ... ff ... , ... [21], ...

Acknowledgements.

A ... , ... ; ... . A ... H ... N ... - ff ... .

B ffi *W* , C ' . J. B
(3.11) L [4].
M K , H H -O .
G.-C. D. L , P . D. E. G
(L) E [28];
M D . F , N ;
L ' *éloge*

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E G fi , L

, A .” L fi . fi

fi L *W*

(Q H)

, , L

[39, 38].

E [28] *On a series of Lambert and some of its significant properties.* I , E L

L ”, *the ingenious engineer Lambert.* A E , A H ,

L , 1764,

L B . A E G , 17 M 1764, .

I [49], L fi

((1.2)). L *Acta Helvetica*

. B E

. E ;

(L) , L ’ . L E 18 O 1771,

E ;

. I E

[O O , 4, 1, 246] A. J. L

F , *éloge* L ’

(1946 1948) fi L

(P ’).

L A L ,

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