Integral Transforms and Special Functions . 00, N . 00, M 2011, 1 13

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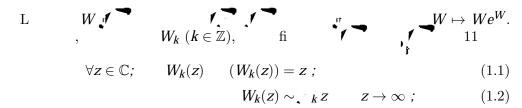
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We show that many functions containing the Lambert W function are Stieltjes functions. We extend the known properties of the set of Stieltjes functions and also prove a generalization of a conjecture of Jackson, Procacci & Sokal. In addition, we consider the relationship of functions of W to the class of completely monotonic functions and show that W is a complete Bernstein function.

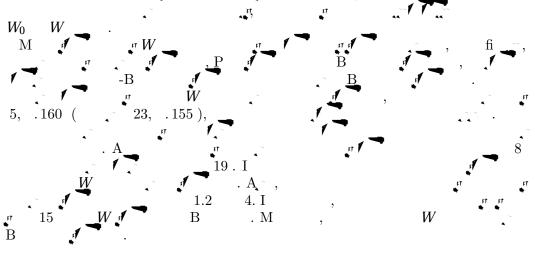
 ${\bf K}_{-j}-{\bf d}$ : LambertWfunction; Stieltjes functions; completely monotonic functions; Bernstein functions; complete Bernstein functions

AMS S b c C a fica : primary 33E99; 30E20; secondary 26A48

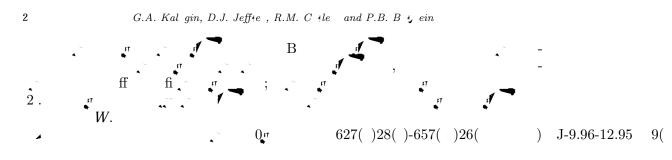
## 1. Introduction



$$k z = \sum_{k=0}^{n} z + 2 \quad ik, \qquad z \qquad \qquad k = 0, \qquad \qquad 14.$$



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$$z, \ldots t = 0, \quad u = u(s) \quad v = v(s)$$

$$u = v \qquad v; \tag{1.11}$$

$$s = s(v) = v$$
 (v) $e^{v \tan v}$ : (1.12)

$$W'(z) = \frac{W(z)}{z(1+W(z))} :$$
(1.13)

LEMMA 1.1 Function W(-t) is nonnegative and bounded on the real line and continuously di erentiable for  $t \neq 1=e$ . Speci cally, it is zero for  $t \in (-\infty; 1=e \text{ and } a \text{ monotone increasing function for } t \in (1=e;\infty) \text{ so that } W(-t) \rightarrow \text{ as } t \rightarrow \infty$ . Correspondingly, the derivative d W(-t)=dt is zero for t < 1=e and positive for t > 1=e. In addition, d W(-t)=dt = o(1=t) as  $t \rightarrow \infty$ .

Proof O  

$$W(-t)$$
  
 $v(t) = W(t)$   
 $v'(t) = \frac{A(v(t))}{t};$   $A(v) = \frac{v}{v^2 + (1 - v - v)^2};$  (1.14)

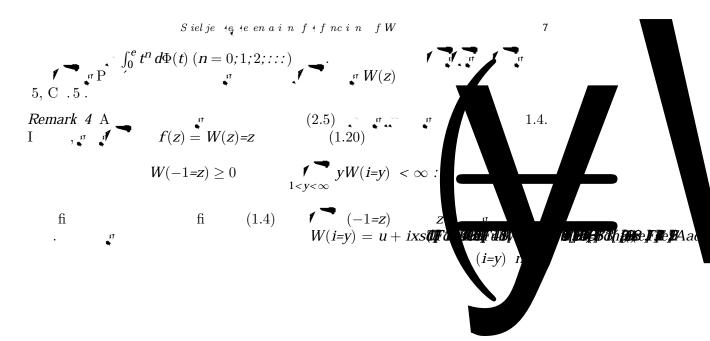
## 1.2. Stieltjes functions



DEFINITION 1.2 A function  $f : (0, \infty) \to \mathbb{R}$  is called a *if it* admits a representation

$$f(x) = a + \int_0^\infty \frac{d(t)}{x+t} \quad (x > 0);$$
(1.15)

where a is a non-negative constant and is a positive measure on  $0;\infty$ ) such that  $\int_0^\infty (1+t)^{-1} d(t) < \infty$ .



8

G.A. Kal gin, D.J. Jeffte , R.M. C tle and P.B. B ; ein

$$0 \le \le 1. \qquad () \qquad c = 1$$

$$g(x) = x \qquad b(x) = 1 + W(x) \in . F \qquad () \qquad a(x)$$

$$b(x). \qquad () \qquad A \qquad () \qquad W(x) = x).$$

$$() \qquad A \qquad () \qquad W(x) = x). \qquad () \qquad c = 1 \qquad g(x) = x \quad (-1 \le \le 0).$$

COROLLARY 2.3 The derivative W'(x) is a Stieltjes function.

Proof 
$$r$$
  $r$   $r$   $()$   $r$  2.2,  $c = 1$ ,  $r$   $(1.13).$ 

THEOREM 2.4 The following functions are Stieltjes functions for each xed real  $a \in (0; e:$ 

$$F_0(z) = \frac{z}{1+z} W(a(1+z)) = W(a(1+z)) - W(a)^2 ; \qquad (2.6)$$

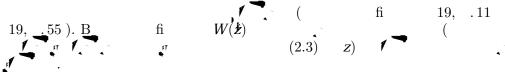
$$F_1(z) = zW\left(\frac{a}{1+z}\right) \left/ \left[W(a) - W\left(\frac{a}{1+z}\right)\right]^2 \right.$$
(2.7)

Proof fi 1.5  $F_0(z)$ .  $F_0(z)$ 

4

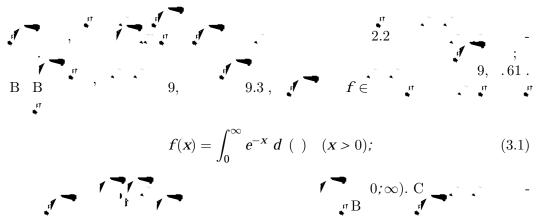
() 
$$t = a$$
.  
 $z = x \in \mathbb{R},$   
 $(t;s) \ t \in \mathbb{R}; s > 0$ .  
 $H(t) = \frac{v}{(b+v-v)^2 + v^{2/2}} \left(\frac{v^2}{2}v - b^2\right) \left(1 - \frac{a}{t}\right) :$   
 $v \in (0; )$ ,  $r \ t \in (-\infty; -1=e),$   
 $v^2 = -2 \ v > 1. \ (0;350.20.909)$ ,  $r1.;$ 

G.A. Kal gin, D.J. Jeffie , R.M. C ile and P.B. B ; ein 10



## 3. Completely monotonic functions

DEFINITION 3.1 A function  $f : (0,\infty) \rightarrow \mathbb{R}$  is called a completely monotonic function if f has derivatives of all orders and satis es  $(-1)^n f^{(n)}(x) \ge 0$  for x > 0, *n* = 0;1;2;:::



DEFINITION 3.2 [8, De nition 5.1] A function  $f: (0,\infty) \rightarrow 0,\infty$  is called a Bernstein function if it is  $C^{\infty}$  and f' is completely monotonic.

 $W' \in \subset$ В 1 гт гт , W• 15ff , *"W*. Γ΄-Κ  $f(\mathbf{x})$ ΑB

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-x}\right) d() ; \qquad (3.2)$$

$$\begin{array}{c} a; b \ge 0 \\ \infty. \ \mathbf{I} \\ \mathbf{r} \\ \mathbf{f}' \\ \mathbf{f}' \\ \mathbf{8} \\ \end{array} \begin{array}{c} \mathbf{L}' \\ \mathbf{f}' \\ \mathbf{8} \\ \mathbf{K}' \end{array} \begin{array}{c} \mathbf{f}' \\ \mathbf{8} \\ \mathbf{K}' \\$$

А 8, 5.4

$$g \in 0 \Rightarrow 1=g \in :$$
 (3.3)

Proof

G.A. Kal gin, D.J. Jeffte , R.M. C the and P.B. B ; ein

$$f \in f(x), \quad x = 0 \quad (f = f), \quad f \in f = g \in f \quad f = g \in g(x) = 0$$

$$f(x) = f(0) + g(0) - g(x) :$$
(4.2)

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$$f \in \Leftrightarrow 1 = f \in 0$$
; (4.3)

$$f \in \Leftrightarrow f(x) = x \in :$$
 (4.4)

## References

[1]