Integral Transforms and Special Functions \cdot 00, N . 00, M 2011, 1–13

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We show that many functions containing the Lambert *W* function are Stieltjes functions. We extend the known properties of the set of Stieltjes functions and also prove a generalization of a conjecture of Jackson, Procacci & Sokal. In addition, we consider the relationship of functions of *W* to the class of completely monotonic functions and show that *W* is a complete Bernstein function.

K_t **d**: Lambert *W* function; Stieltjes functions; completely monotonic functions; Bernstein functions; complete Bernstein functions

AMS S b c C a fica : primary 33E99; 30E20; secondary 26A48

1. Introduction

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\kappa z = \kappa z + 2 \, ik, \qquad z \qquad k = 0, \qquad l = 14.
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\sum_{i=1}^{3} \mathcal{N}_1 = -24 \quad \text{R} \quad \text{F} \quad \text
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S\;iel\;je\; \; \mathfrak{t}\mathfrak{e}\; \; \mathfrak{t}\mathfrak{e}\; en\; a\; i\; n\; f\; \mathfrak{t}\; f\; nc\; i\; n\; \;f\; W \qquad \qquad 3
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\mathbf{F} \qquad \qquad \mathbf{F} \qquad \qquad \mathbf{Z}, \qquad t = 0, \qquad \mathbf{F} \qquad \qquad \mathbf{u} = \mathbf{u}(\mathbf{s}) \qquad \mathbf{v} = \mathbf{v}(\mathbf{s})
$$

$$
u = v \qquad v; \tag{1.11}
$$

$$
s = s(v) = v \qquad (v)e^{v\tan v} \tag{1.12}
$$

$$
\cdot^{\pi}W(z)
$$

$$
W'(z) = \frac{W(z)}{z(1 + W(z))} \tag{1.13}
$$

$$
\mathbb{F}^{\bullet} \quad , \quad \mathbb{F}^{\bullet} \quad , \qquad \mathbb{F}^{\bullet
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Lemma 1.1 *Function ^W*(*−t*) *is nonnegative and bounded on the real line and continuously differentiable for* $t \neq 1$ =e. Specifically, it is zero for $t \in (-\infty, 1=e$ and *a* monotone increasing function for $t \in (1=e,\infty)$ *so that* $W(-t) \to as t \to \infty$ *. Correspondingly, the derivative d* $W(-t) = dt$ *is zero for* $t < 1 = e$ *and positive for* $t > 1 = e$ *. In addition, d* $W(-t) = dt = o(1=t)$ *as* $t \to \infty$ *.*

Proof O
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W(-t)
$$
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$$
v(t) = W(t)
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$$
V(t) = \frac{A(v(t))}{t}; \quad A(v) = \frac{v}{v^2 + (1 - v - v)^2}.
$$
\n(1.14)

Then the derivative *d W*(*−t*)*=dt* = *A*(*v*(*−t*))*=t*, which implies that it is zero for *t <* 1*=e* and positive for *t >* 1*=e* as *v*(*t*) = 0 for *t > −*1*=e* and *v*(*t*) *>* 0 for *t < −*1*=e*. It remains to justify the estimation of the derivative *d W*(*−t*)*=dt* at large *t* but it immediately follows from the two facts that *v*(*−t*) *→* as *t → ∞* and that *A*(*v*) *→* 0 as *v →* .

1.2. *Stieltjes functions*

DEFINITION 1.2 *A* function $f : (0, \infty) \to \mathbb{R}$ is called a *admits a representation*

$$
f(x) = a + \int_0^\infty \frac{d(t)}{x + t} \quad (x > 0); \tag{1.15}
$$

where a is a non-negative constant and is a positive measure on 0 ; ∞) *such that* $\int_0^\infty (1 + t)^{-1} d(t) < \infty$.

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g(x) = x \qquad \qquad \downarrow
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b(x).
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COROLLARY 2.3 *The derivative* $W'(x)$ *is a Stieltjes function.*

Proof
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\int_{a}^{\pi} \int_{1}^{\pi} (1.13). \qquad (1.13)
$$

THEOREM 2.4 *The following functions are Stieltjes functions for each xed real a ∈* (0*; e :*

$$
F_0(z) = \frac{z}{1+z}W(a(1+z)) = W(a(1+z)) - W(a)^2 \quad ; \tag{2.6}
$$

$$
F_1(z) = zW\left(\frac{a}{1+z}\right) / \left[W(a) - W\left(\frac{a}{1+z}\right)\right]^2 \tag{2.7}
$$

Proof λ first apply the function \mathcal{F} **F**₀(*z*). $F_0(z)$

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S \text{ det } \phi \cdot q \cdot \phi \text{ or } \phi \text{ if } q \neq r \text{ and } q \text{ if } q \
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3. Completely monotonic functions

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\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \right| \right| \right| \, d\mu \right| \, d\mu
$$

DEFINITION 3.1 *A function* $f : (0, \infty) \rightarrow \mathbb{R}$ *is called a completely monotonic function if f has derivatives of all orders and satis es* $(-1)^n f^{(n)}(x) \ge 0$ *for* $x > 0$ *,* $n = 0; 1; 2; ...$

DEFINITION 3.2 *[8, De nition 5.1] A function* $f : (0, \infty) \rightarrow 0, \infty)$ *is called a Bernstein function if it is* C^{∞} *and* f' *is completely monotonic.*

Since *W′ ∈ S ⊂ CM*, *W* is a Bernstein function. The same fact has been esthat in 15 in a different way, based on the polynomials of the poly appearing in W . A B $f(x)$ and $f(x)$ and $f(x)$

$$
f(x) = a + bx + \int_0^\infty \left(1 - e^{-x}\right) d\ (\)\ ;\tag{3.2}
$$

where *a; b [≥]* 0 and is a positive measure on (0*; [∞]*) satisfying [∫] *[∞]* 0 (1+) *[−]*1*d*() *< ∞*. It is called the L´evy measure. The equation (3.2) is obtained by integrating (3.1) written for *f ′* [8].

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\bullet \\
\end{array}
$$

$$
g \in 0 \Rightarrow 1 = g \in \mathfrak{g} \tag{3.3}
$$

() $X W(x) \ (x > 0; \leq -1).$
() $X W (x) [1 + W(x)]$ $(x > 0; \; ; \; \geq 0; -1 \leq \leq 0; \leq 0).$
() $X W (x^{-}) [1 + W(x^{-})]$ $(x > 0; \; ; \; \leq 0; -1 \leq \leq 0; \leq 0).$
() $1 - x^{-}$ $W (x) 1 + W(x)^{-1} (x > 0; 0 \leq \leq 1; -1 \leq \leq 0; 0 \leq \leq 1).$

Proof

$$
(\) \qquad W(x)=x \in \mathbb{C} \longrightarrow x \in \mathbb{C}, \qquad x \in \mathbb{C} \longrightarrow x \
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\mathbf{f} \bullet \mathbf{f} \qquad \qquad \mathbf{f} \in \mathbb{R}^{n} \text{ and } \mathbf{f} \in \mathbb{R}^{n} \text{ and } \mathbf{f} \bullet \mathbf{g} \in \mathbb{R}^{n} \text{ and } \mathbf{g} \in \mathbb{R}^{n} \text{ and } \mathbf{g} \in \mathbb{R}^{n} \text{ and } \mathbf{g} \text{ and } \mathbf{g} \text{ and } \mathbf{g} \text{ are } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{ is the same as } \mathbf{f} \bullet \mathbf{g} \text{ and } \mathbf{g} \text{
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$$
f(x) = f(0) + g(0) - g(x) : \t\t(4.2)
$$

 $1, 19, 7.3, 19, 6.2(), (.)$

$$
f \in \Leftrightarrow 1 = f \in \ldots \quad (4.3)
$$

$$
f \in \Leftrightarrow f(x)=x \in \ldots \tag{4.4}
$$

We note at once that the statement (4.3) together with that 1*=W ∈ S* (by Theorem 2.2(a) with *c* = 0) immediately results in a conclusion that *W* is a complete Bernstein function. Now we go back to the properties of the set listed in Section 1.2 to prove the last three properties therein. Let *f ∈ S \ {*0 . (x) Apply sequentially (vii), (4.1), (4.3), (i), to obtain *f ∈ S* (0 *≤ ≤* 1) *⇒ ^f* (0) *[−] ^f* (*x*) *∈ CB ⇒ ^g*(*x*) = [*^f* (0) *[−] ^f* (*x*)]*−*¹ *∈ S ⇒* ¹*=g*(1*=x*) = *^f* (0) *[−] f* (1*=x*) *∈ S*; (xi) Apply sequentially (4.1), (4.3), (ii), to obtain *f*(0) *− f*(*x*) *∈ CB ⇒ g*(*x*) =

^f(0) *[−] ^f*(*x*)]*−*¹ *∈ S ⇒* ¹*=*(*xg*(*x*)) = (*f*(0)*−f*(*x*))*=x ∈ S ⇒* (1*−f*(*x*)*=f*(0))*=x ∈ S*; (xii) By (vii), *f ∈ S* (0 *≤ ≤* 1). Suppose that lim*x→*⁰ *f*(*x*) = *b ≤ ∞* and lim*x→∞ f*(*x*) = *c* where 0 *< c < ∞*. Then *b [−] ≤ f [−] ≤ c −* , i.e. *f −* is bounded. In addition, *f [−] ∈ CB* by (4.3). Therefore the statement (4.2) can be applied, i.e. there exists a bounded function *g ∈ S;* lim*x→∞ g*(*x*) = 0 such that we can write *g*(*x*) = *g*(0)+*b [−] −f [−]* (*x*). Taking the last equation in the limit *x → ∞* we obtain *g*(0) + *b −* = *c −* , hence *g* = *c [−] − f −*

References

[1]