Integral Transforms and Special Functions
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We show that many functions containing the Lambert W function are Stieltjes functions. We extend the known properties of the set of Stieltjes functions and also prove a generalization of a conjecture of Jackson, Procacci & Sokal. In addition, we consider the relationship of functions of W to the class of completely monotonic functions and show that W is a complete Bernstein function.

K d : Lambert W function; Stieltjes functions; completely monotonic functions; Bernstein functions; complete Bernstein functions

AMS S b c C a fica : primary 33E99; 30E20; secondary 26A48

1. Introduction

L W $W \mapsto We^W$. , W_k $(k \in \mathbb{Z})$, fi 11

$$\forall z \in \mathbb{C}; \qquad W_k(z) \qquad (W_k(z)) = z \ ; \tag{1.1}$$

$$W_k(z) \sim \ _k z \qquad z \to \infty \ ; \tag{1.2}$$

$_k z = \ z + 2 \ ik,$ z 14 . $k = 0,$

W_0 W .
M W , fi , P B , -B B . W , 5, .160 (23, .155), . . A 8 19 . I W . A , 1.2 4. I , 15 W B . M , W B .

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F z, . . $t = 0$, $u = u(s)$ $v = v(s)$

$$u = v \quad v; \tag{1.11}$$

$$s = s(v) = v \quad (v)e^{v \tan v} : \tag{1.12}$$

$W(z)$

$$W'(z) = \frac{W(z)}{z(1 + W(z))} : \tag{1.13}$$

F , .

LEMMA 1.1 *Function* $W(-t)$ *is nonnegative and bounded on the real line and continuously di erentiable for* $t \neq 1{=}e$*. Specifically, it is zero for* $t \in (-\infty; 1{=}e$ *and a monotone increasing function for* $t \in (1{=}e; \infty)$ *so that* $W(-t) \to$ *as* $t \to \infty$*. Correspondingly, the derivative* $d \; W(-t){=}dt$ *is zero for* $t < 1{=}e$ *and positive for* $t > 1{=}e$*. In addition,* $d \; W(-t){=}dt = o(1{=}t)$ *as* $t \to \infty$*.*

Proof O $W(t)$ ((1.3)), $W(-t)$ t $W(-t) \to$ $t \to \infty$. ff $t = 1{=}e$. $v(t) = \; W(t)$ $v'(t)$; (1.13) (1.9)

$$v'(t) = \frac{A(v(t))}{t}; \quad A(v) = \frac{v}{v^2 + (1 - v \quad v)^2} : \tag{1.14}$$

$d \; W(-t){=}dt = A(v(-t)){=}t$, $t < 1{=}e$ $t > 1{=}e$ $v(t) = 0$ $t > -1{=}e$ $v(t) > 0$ $t < -1{=}e$. I $d \; W(-t){=}dt$ t $v(-t) \to$ $t \to \infty$ $A(v) \to 0$ $v \to$. ∎

1.2. Stieltjes functions

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DEFINITION 1.2 *A function* $f : (0; \infty) \to \mathbb{R}$ *is called a* *if it admits a representation*

$$f(x) = a + \int_0^\infty \frac{d \;\, (t)}{x + t} \quad (x > 0); \tag{1.15}$$

where a *is a non-negative constant and* *is a positive measure on* $0; \infty)$ *such that* $\int_0^\infty (1 + t)^{-1} d \;\, (t) < \infty$*.*

A *Stieltjes transform* 9, . 127 . E 2.3 , fi (1.15).

fi $a \geq 0$ $1 = x$

$\int_0^e t^n \, d\Phi(t)$ $(n = 0; 1; 2; \ldots)$.

P $W(z)$

5, C . 5 .

Remark 4 A (2.5) 1.4.

I , $f(z) = W(z)=z$ (1.20)

$$W(-1=z) \geq 0 \qquad \sup_{1<y<\infty} yW(i=y) < \infty :$$

fi fi (1.4) $(-1=z)$ z

. $W(i=y) = u + ixs$

$(i=y)$ n

$0 \leq \quad \leq 1$. ( ) ( ) $c = 1$
$g(x) = x$ $b(x) = 1 + W(x\ ) \in$. F ( ) $a(x)$
$b(x)$.

( ) A ( ) ( ) $= -1$ (1.7)
(( ) $W(x)/x$).

( ) A ( ) (( )) ( ) (( ))
( ) $c = 1$ $g(x) = x$ $(-1 \leq \quad \leq 0)$.

■

Corollary 2.3 *The derivative $W'(x)$ is a Stieltjes function.*

Proof ( ) 2.2, $c = 1$,
(1.13). ■

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Theorem 2.4 *The following functions are Stieltjes functions for each fixed real $a \in (0; e$:*

$$F_0(z) = \frac{z}{1+z} W(a(1+z))/ \ W(a(1+z)) - W(a)\ ^2 \ ; \tag{2.6}$$

$$F_1(z) = zW\left(\frac{a}{1+z}\right) \Big/ \left[W(a) - W\left(\frac{a}{1+z}\right)\right]^2 \ : \tag{2.7}$$

Proof fi 1.5 $F_0(z)$.
$F_0(z)$

( ) $t = a$. ( ). $V(z)$ ( )

$z = x \in \mathbb{R}$, (2.9)

$(t; s)$ $t \in \mathbb{R}; s > 0$. (1.9)

$$H(t) = \frac{v}{(b + v \quad v)^2 + v^2 \; {}^2} \left(\frac{v^2}{{}^2 v} - b^2 \right) \left(1 - \frac{a}{t} \right) :$$

$v \in (0; \;)$ $t \in (-\infty; -1/e)$, $v^2 = \; {}^2 v > 1$. (0;350.20.909 1.;

t

$(t$

(fi 19, . 11 19, . 55). B fi $W(z)$ ((2.3) z) .

3. Completely monotonic functions

, fi 3 .

Definition 3.1 *A function* $f : (0;\infty) \to \mathbb{R}$ *is called a completely monotonic function if* f *has derivatives of all orders and satisfies* $(-1)^n f^{(n)}(x) \geq 0$ *for* $x > 0$, $n = 0; 1; 2; \ldots$

, 2.2 - . ; 9, . 61 . B B ' 9, 9.3 , $f \in$

$$f(x) = \int_0^\infty e^{-x} \, d \; (\;) \quad (x > 0); \tag{3.1}$$

0; ∞). C - B .

Definition 3.2 *[8, Definition 5.1] A function* $f : (0;\infty) \to 0;\infty)$ *is called a Bernstein function if it is* C^∞ *and* f' *is completely monotonic.*

$W' \in \subset$, W B . - 15 ff , W . A B $f(x)$ L´ -K

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-x}\right) d \; (\;) \; ; \tag{3.2}$$

$a; b \geq 0$ $(0;\infty)$ $\int_0^\infty (1+ \;)^{-1} d \; (\;) <$ ∞. I L´ . (3.2) (3.1) f' 8 . A 8, 5.4

$$g \in \quad 0 \Rightarrow 1/g \in \; : \tag{3.3}$$

( ) $x\ W(x)$ $(x > 0;\ \ \le -1)$.
( ) $x\ W\ (x\ )\left[1 + W(x\ )\right]$ $(x > 0;\ ;\ \ge 0; -1 \le\ \ \le 0;\ \ \le 0)$.
( ) $x\ W\ (x^{-}\ )\left[1 + W(x^{-}\ )\right]$ $(x > 0;\ ;\ \le 0; -1 \le\ \ \le 0;\ \ \le 0)$.
( ) $1 - x^{-}\ \ W\ \ (x\ )\ 1 + W(x\ )\ ^{-1}$ $(x > 0;\ 0 \le\ \ \le 1; -1 \le\ \ \le 0;\ 0 \le\ \ \le 1)$.

Proof

( ) $W(x)/x \in\ \ \subset$ $x \in$ ≤ 0, $x\ W(x)$ $(\ \le -1)$ ( ) .

( ) $f\ (x) = x^{-}\ \in$ $(x > 0;\ \ge 0)$ $g(x) = 1/W(x\ )$ $h(x) = 1/(1 + W(x\ ))$ $-1 \le\ \ \le 0$. $1/g \in$ $1/h \in$ 2.2 ( ) $c = 0$ $c = 1$, L 3.3 $f\ (g(x)) = g^{-}\ (x) \in$ $f\ (h(x)) = h^{-}\ (x) \in$ $(\ \ge 0)$. $g(x)$ $h(x)$ $x \in$ $(x > 0;\ \le 0)$, ( ) .

( ) C $f\ (x) = x\ \in$ $(x > 0;\ \le 0)$ $g(x$

$f(x)$ $x = 0$ ( ) fi . A f
$f \in$, $g \in$ $_{x\to\infty} g(x) = 0$

$$f(x) = f(0) + g(0) - g(x) :\qquad (4.2)$$

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$$f \in \quad \Leftrightarrow 1/f \in \quad 0 \quad ;\qquad (4.3)$$

$$f \in \quad \Leftrightarrow f(x)/x \in \quad :\qquad (4.4)$$

(4.3) $1/W \in$ (-
2.2( ) $c = 0$) W
B .
N 1.2
. L $f \in$ 0 .
( ) A ( ), (4.1), (4.3), ( ), $f \in$ (0 ≤ ≤ 1) ⇒
f (0) − f (x) ∈ ⇒ $g(x)$ = f (0) − f (x) $^{-1}$ ∈ ⇒ $1/g(1/x)$ = f (0) −
f ($1/x$) ∈ ;
( ) A (4.1), (4.3), ( ), $f(0) - f(x) \in$ ⇒ $g(x)$ =
$f(0) - f(x)$ $^{-1}$ ∈ ⇒ $1/(xg(x)) = (f(0) - f(x))/x \in$ ⇒ $(1 - f(x)/f(0))/x \in$;
( ) B ( ), f ∈ (0 ≤ ≤ 1). $_{x\to 0} f(x) = b \le \infty$
$_{x\to\infty} f(x) = c$ $0 < c < \infty$. b^{-} ≤ f^{-} ≤ c^{-} , . . f^{-} .
I , f^{-} ∈ (4.3). (4.2) , . .
$g \in$; $_{x\to\infty} g(x) = 0$
$g(x) = g(0) + b^{-} - f^{-}(x)$. $x \to \infty$
$g(0) + b^{-} = c^{-}$, $g = c^{-} - f^{-}$

References

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