

The asymptotic approach of two parallel spheres has been studied by Batchelor (1969), Hansford (1970) and Jeffrey (1982) with the aim of calculating asymptotically the forces exerted by the spheres on the fluid. These forces are not the only quantities of interest, however, because the analysis of the properties of suspensions of spheres requires also the stresses of the spheres, defined by

$$- \int_A \mathbf{x}' \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, dA, \quad (1.3)$$

where the vector  $\mathbf{x}'$  is drawn from the centre of the sphere. Various authors have defined resistance





$$H = 0 \quad W = R_0 + \frac{1}{2} \epsilon^2 R_2 + \frac{1}{2} \epsilon^4 R_4 + \dots + \epsilon^2 K^* K.$$

Since

$$R_0(H_1) + \epsilon U_1(K, H_1) + \epsilon^2 K^* \partial U_1 / \partial Z|_{Z=H_1},$$

$$U(R_0 Z - H_1 + \epsilon^2 K^* K) = U_1(Z),$$

we can equate powers of  $\epsilon$  to find

$$R_0(H_1) = U_1(H_1), \quad (2.10)$$

$$U_1(K, H_1) + \epsilon^2 K^* \partial U_1 / \partial Z|_{Z=H_1} = 0, \quad (2.11)$$

$$R_2(H_1) + \epsilon U_2(K, H_1) + \epsilon^2 K^* \partial U_2 / \partial Z|_{Z=H_1} + \epsilon^2 U_1(K, H_1) = 0, \quad (2.12)$$

$$R_4(H_1) + \epsilon U_4(K, H_1) + \epsilon^2 K^* \partial U_4 / \partial Z|_{Z=H_1} + \epsilon^2 U_2(K, H_1) + \epsilon^4 U_1(K, H_1) = 0, \quad (2.13)$$

$$R_6(H_1) + \epsilon U_6(K, H_1) + \epsilon^2 K^* \partial U_6 / \partial Z|_{Z=H_1} + \epsilon^2 U_4(K, H_1) + \epsilon^4 U_2(K, H_1) + \epsilon^6 U_1(K, H_1) = 0, \quad (2.14)$$

$$R_8(H_1) + \epsilon U_8(K, H_1) + \epsilon^2 K^* \partial U_8 / \partial Z|_{Z=H_1} + \epsilon^2 U_6(K, H_1) + \epsilon^4 U_4(K, H_1) + \epsilon^6 U_2(K, H_1) + \epsilon^8 U_1(K, H_1) = 0, \quad (2.15)$$

$$R_{10}(H_1) + \epsilon U_{10}(K, H_1) + \epsilon^2 K^* \partial U_{10} / \partial Z|_{Z=H_1} + \epsilon^2 U_8(K, H_1) + \epsilon^4 U_6(K, H_1) + \epsilon^6 U_4(K, H_1) + \epsilon^8 U_2(K, H_1) + \epsilon^{10} U_1(K, H_1) = 0, \quad (2.16)$$

$$R_{12}(H_1) + \epsilon U_{12}(K, H_1) + \epsilon^2 K^* \partial U_{12} / \partial Z|_{Z=H_1} + \epsilon^2 U_{10}(K, H_1) + \epsilon^4 U_8(K, H_1) + \epsilon^6 U_6(K, H_1) + \epsilon^8 U_4(K, H_1) + \epsilon^{10} U_2(K, H_1) + \epsilon^{12} U_1(K, H_1) = 0, \quad (2.17)$$

$$R_{14}(H_1) + \epsilon U_{14}(K, H_1) + \epsilon^2 K^* \partial U_{14} / \partial Z|_{Z=H_1} + \epsilon^2 U_{12}(K, H_1) + \epsilon^4 U_{10}(K, H_1) + \epsilon^6 U_8(K, H_1) + \epsilon^8 U_6(K, H_1) + \epsilon^{10} U_4(K, H_1) + \epsilon^{12} U_2(K, H_1) + \epsilon^{14} U_1(K, H_1) = 0, \quad (2.18)$$

$$R_{16}(H_1) + \epsilon U_{16}(K, H_1) + \epsilon^2 K^* \partial U_{16} / \partial Z|_{Z=H_1} + \epsilon^2 U_{14}(K, H_1) + \epsilon^4 U_{12}(K, H_1) + \epsilon^6 U_{10}(K, H_1) + \epsilon^8 U_8(K, H_1) + \epsilon^{10} U_6(K, H_1) + \epsilon^{12} U_4(K, H_1) + \epsilon^{14} U_2(K, H_1) + \epsilon^{16} U_1(K, H_1) = 0, \quad (2.19)$$

$$R_{18}(H_1) + \epsilon U_{18}(K, H_1) + \epsilon^2 K^* \partial U_{18} / \partial Z|_{Z=H_1} + \epsilon^2 U_{16}(K, H_1) + \epsilon^4 U_{14}(K, H_1) + \epsilon^6 U_{12}(K, H_1) + \epsilon^8 U_{10}(K, H_1) + \epsilon^{10} U_8(K, H_1) + \epsilon^{12} U_6(K, H_1) + \epsilon^{14} U_4(K, H_1) + \epsilon^{16} U_2(K, H_1) + \epsilon^{18} U_1(K, H_1) = 0, \quad (2.20)$$

$$R_{20}(H_1) + \epsilon U_{20}(K, H_1) + \epsilon^2 K^* \partial U_{20} / \partial Z|_{Z=H_1} + \epsilon^2 U_{18}(K, H_1) + \epsilon^4 U_{16}(K, H_1) + \epsilon^6 U_{14}(K, H_1) + \epsilon^8 U_{12}(K, H_1) + \epsilon^{10} U_{10}(K, H_1) + \epsilon^{12} U_8(K, H_1) + \epsilon^{14} U_6(K, H_1) + \epsilon^{16} U_4(K, H_1) + \epsilon^{18} U_2(K, H_1) + \epsilon^{20} U_1(K, H_1) = 0, \quad (2.21)$$

$$R_{22}(H_1) + \epsilon U_{22}(K, H_1) + \epsilon^2 K^* \partial U_{22} / \partial Z|_{Z=H_1} + \epsilon^2 U_{20}(K, H_1) + \epsilon^4 U_{18}(K, H_1) + \epsilon^6 U_{16}(K, H_1) + \epsilon^8 U_{14}(K, H_1) + \epsilon^{10} U_{12}(K, H_1) + \epsilon^{12} U_{10}(K, H_1) + \epsilon^{14} U_8(K, H_1) + \epsilon^{16} U_6(K, H_1) + \epsilon^{18} U_4(K, H_1) + \epsilon^{20} U_2(K, H_1) + \epsilon^{22} U_1(K, H_1) = 0, \quad (2.22)$$

$$R_{24}(H_1) + \epsilon U_{24}(K, H_1) + \epsilon^2 K^* \partial U_{24} / \partial Z|_{Z=H_1} + \epsilon^2 U_{22}(K, H_1) + \epsilon^4 U_{20}(K, H_1) + \epsilon^6 U_{18}(K, H_1) + \epsilon^8 U_{16}(K, H_1) + \epsilon^{10} U_{14}(K, H_1) + \epsilon^{12} U_{12}(K, H_1) + \epsilon^{14} U_{10}(K, H_1) + \epsilon^{16} U_8(K, H_1) + \epsilon^{18} U_6(K, H_1) + \epsilon^{20} U_4(K, H_1) + \epsilon^{22} U_2(K, H_1) + \epsilon^{24} U_1(K, H_1) = 0, \quad (2.23)$$

$$R_{26}(H_1) + \epsilon U_{26}(K, H_1) + \epsilon^2 K^* \partial U_{26} / \partial Z|_{Z=H_1} + \epsilon^2 U_{24}(K, H_1) + \epsilon^4 U_{22}(K, H_1) + \epsilon^6 U_{20}(K, H_1) + \epsilon^8 U_{18}(K, H_1) + \epsilon^{10} U_{16}(K, H_1) + \epsilon^{12} U_{14}(K, H_1) + \epsilon^{14} U_{12}(K, H_1) + \epsilon^{16} U_{10}(K, H_1) + \epsilon^{18} U_8(K, H_1) + \epsilon^{20} U_6(K, H_1) + \epsilon^{22} U_4(K, H_1) + \epsilon^{24} U_2(K, H_1) + \epsilon^{26} U_1(K, H_1) = 0, \quad (2.24)$$

$$R_{28}(H_1) + \epsilon U_{28}(K, H_1) + \epsilon^2 K^* \partial U_{28} / \partial Z|_{Z=H_1} + \epsilon^2 U_{26}(K, H_1) + \epsilon^4 U_{24}(K, H_1) + \epsilon^6 U_{22}(K, H_1) + \epsilon^8 U_{20}(K, H_1) + \epsilon^{10} U_{18}(K, H_1) + \epsilon^{12} U_{16}(K, H_1) + \epsilon^{14} U_{14}(K, H_1) + \epsilon^{16} U_{12}(K, H_1) + \epsilon^{18} U_{10}(K, H_1) + \epsilon^{20} U_8(K, H_1) + \epsilon^{22} U_6(K, H_1) + \epsilon^{24} U_4(K, H_1) + \epsilon^{26} U_2(K, H_1) + \epsilon^{28} U_1(K, H_1) = 0, \quad (2.25)$$

$$R_{30}(H_1) + \epsilon U_{30}(K, H_1) + \epsilon^2 K^* \partial U_{30} / \partial Z|_{Z=H_1} + \epsilon^2 U_{28}(K, H_1) + \epsilon^4 U_{26}(K, H_1) + \epsilon^6 U_{24}(K, H_1) + \epsilon^8 U_{22}(K, H_1) + \epsilon^{10} U_{20}(K, H_1) + \epsilon^{12} U_{18}(K, H_1) + \epsilon^{14} U_{16}(K, H_1) + \epsilon^{16} U_{14}(K, H_1) + \epsilon^{18} U_{12}(K, H_1) + \epsilon^{20} U_{10}(K, H_1) + \epsilon^{22} U_8(K, H_1) + \epsilon^{24} U_6(K, H_1) + \epsilon^{26} U_4(K, H_1) + \epsilon^{28} U_2(K, H_1) + \epsilon^{30} U_1(K, H_1) = 0, \quad (2.26)$$

$$R_{32}(H_1) + \epsilon U_{32}(K, H_1) + \epsilon^2 K^* \partial U_{32} / \partial Z|_{Z=H_1} + \epsilon^2 U_{30}(K, H_1) + \epsilon^4 U_{28}(K, H_1) + \epsilon^6 U_{26}(K, H_1) + \epsilon^8 U_{24}(K, H_1) + \epsilon^{10} U_{22}(K, H_1) + \epsilon^{12} U_{20}(K, H_1) + \epsilon^{14} U_{18}(K, H_1) + \epsilon^{16} U_{16}(K, H_1) + \epsilon^{18} U_{14}(K, H_1) + \epsilon^{20} U_{12}(K, H_1) + \epsilon^{22} U_{10}(K, H_1) + \epsilon^{24} U_8(K, H_1) + \epsilon^{26} U_6(K, H_1) + \epsilon^{28} U_4(K, H_1) + \epsilon^{30} U_2(K, H_1) + \epsilon^{32} U_1(K, H_1) = 0, \quad (2.27)$$

$$R_{34}(H_1) + \epsilon U_{34}(K, H_1) + \epsilon^2 K^* \partial U_{34} / \partial Z|_{Z=H_1} + \epsilon^2 U_{32}(K, H_1) + \epsilon^4 U_{30}(K, H_1) + \epsilon^6 U_{28}(K, H_1) + \epsilon^8 U_{26}(K, H_1) + \epsilon^{10} U_{24}(K, H_1) + \epsilon^{12} U_{22}(K, H_1) + \epsilon^{14} U_{20}(K, H_1) + \epsilon^{16} U_{18}(K, H_1) + \epsilon^{18} U_{16}(K, H_1) + \epsilon^{20} U_{14}(K, H_1) + \epsilon^{22} U_{12}(K, H_1) + \epsilon^{24} U_{10}(K, H_1) + \epsilon^{26} U_8(K, H_1) + \epsilon^{28} U_6(K, H_1) + \epsilon^{30} U_4(K, H_1) + \epsilon^{32} U_2(K, H_1) + \epsilon^{34} U_1(K, H_1) = 0, \quad (2.28)$$



$$y^{(n)}(z) = (1+z)^{-4} \left[ \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} \frac{6\lambda^2}{(1+z)^2} + \frac{(-1)^{n-2}}{(n-2)!} \frac{6\lambda^2(\lambda^2 + 12\lambda - 4)}{(1+z)^4} \right] + O(1) \quad (9)$$

References in passing concerns by using these results were obtained using computer algebra systems.

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conditions into

stretched coordinates, we obtain:

$$W = -1 + \alpha(1-z) \quad \text{and} \quad U = -\frac{1}{2}e^{1/2}R.$$

Therefore, the boundary conditions on the deforming sphere become

$$W(R, H) = 1 \quad \text{and} \quad W(R, H) = R^{-1/2}H^{3/2}.$$

are the same as in the previous

section. The method of solution is also the same and

again the pressure, where  $R$  is large

goes as  $O(R^{-2}) + eO(R^{-2})$ . Since this flow is not

has not been solved before, we have

the stresslet to calculate. However,

the force calculation does not give

any new results, because the reciprocal of the

shows that the forces are proportional to

the resistance matrix is symmetric

Similar results to  $Y_{10}$  are the same as those on  $Y_{10}$ , but we noticed that an overall bias is present in the  $Y_{10}$  results. This is shown in Figure 10.





$n$	without $g_{\infty}$	with $g_{\infty}$	without $g_{\infty}$	with $g_{\infty}$
100	0.716327	0.71631	-0.14194	-0.14493
200	0.71636	0.71636	-0.14424	-0.14574