Approximate solutions to a parameterized sixth order boundary value problem

Songxin Liang [¤], David J. Je®rey

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada, N6A 5B7

Abstract

In this paper, the homotopy analysis method (HAM) is applied to solve a parameterized sixth order boundary value problem which, for large parameter values, cannot be solved by other analytical methods for ¯nding approximate series solutions. Convergent series solutions are obtained, no matter how large the value of the parameter is.

Key words: Homotopy analysis method, boundary value problem, analytical solution, symbolic computation

1 Introduction

Boundary value problems arise in engineering, applied mathematics and several branches of physics, and have attracted much attention. However, it is di±cult to obtain closed-form solutions for boundary value problems, especially for nonlinear problems. In most cases, only approximate solutions (either numerical solutions or analytical solutions) can be expected. Some numerical methods such as ¯nite di®erence method [1], ¯nite element method [2] and shooting method [3] have been developed for obtaining approximate solutions to boundary value problems.

Perturbation method is one of the well-known methods for solving nonlinear problems analytically. However, it strongly depends on the existence of

¤ Corresponding author.
Email address: `sliang22@uwo.ca, Fax: 1(519)661-3523, Tel: 1(519)434-9410` (Songxin Liang).

small/large parameters. Traditional non-perturbation methods such as Adomian's decomposition method [5], di®erential transformation method [6,7] and homotopy perturbation method [8] have been developed for solving boundary value problems. However, these methods have their obvious disadvantages.

Consider the following special sixth order boundary value problem involving a parameter c [8]:

$$u^{(6)}(x) = (1+c)\,u^{(4)}(x) - c\,u''(x) + c\,x, \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} &u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 0, \\ &u(1) = \frac{7}{6} + \sinh(1), \; u'(1) = \frac{1}{2} + \cosh(1), \; u''(1) = 1 + \sinh(1). \end{aligned} \tag{2}$$

The boundary value problem (1,2) is interesting because its exact solution

$$u_{exact}(x) = 1 + \frac{1}{6}x^3 + \sinh(x) \tag{3}$$

does not depend on the parameter c although itself does. This can be explained if we rewrite (1) in the following equivalent form

$$\{u^{(6)}(x) - u^{(4)}(x)\} - c\{u^{(4)}(x) - u''(x) + x\} = 0. \tag{4}$$

lies in the fact that the HAM provides a convenient way to adjust and control the convergence region and rate of the series solutions obtained.

2 Solutions of the problem

We first construct a zeroth order deformation equation

$$(1 - p)L[\phi(x;p) - u_0(x)] = p\,\hbar\,N[\phi(x;p)], \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a convergence-control parameter, and $\phi(x;p)$ is an unknown function, respectively. According to (1), the auxiliary linear operator is given by

$$L[\phi(x;p)] = \frac{\partial^6 \phi(x;p)}{\partial x^6}, \tag{6}$$

and the nonlinear operator is given by

$$N[\phi(x;p)] = \frac{\partial^6 \phi(x;p)}{\partial x^6} - (1+c)\frac{\partial^4 \phi(x;p)}{\partial x^4} + c\frac{\partial^2 \phi(x;p)}{\partial x^2} - cx. \tag{7}$$

Now suppose the initial guess of the solution is of the form

$$u_0(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0. \tag{8}$$

Then using the boundary conditions (2) gives a system of six linear equations in six parameters $a_0, a_1, \ldots, a_5$. Solving the resulting system gives the initial guess

$$\begin{aligned} u_0(x) = x^6 &- \frac{(19 + 24\,\mathrm{e} - 7\,\mathrm{e}^2)\,x^5}{4\,\mathrm{e}} + \frac{(23 + 22\,\mathrm{e} - 9\,\mathrm{e}^2)\,x^4}{2\,\mathrm{e}} \\ &- \frac{(87 + 82\,\mathrm{e} - 39\,\mathrm{e}^2)\,x^3}{12\,\mathrm{e}} + x + 1. \end{aligned} \tag{9}$$

The boundary conditions to (5) can be set as

$$\begin{aligned} &\phi(0;p) = 1, \quad \frac{\partial \phi(0;p)}{\partial x} = 1, \quad \frac{\partial^2 \phi(0;p)}{\partial x^2} = 0, \quad \phi(1;p) = \frac{7}{6} + \sinh(1), \\ &\frac{\partial \phi(1;p)}{\partial x} = \frac{1}{2} + \cosh(1), \quad \frac{\partial^2 \phi(1;p)}{\partial x^2} = 1 + \sinh(1). \end{aligned} \tag{10}$$

We now focus on how to obtain higher order approximations to the problem (1,2). From (5), when $p = 0$ and $p = 1$,

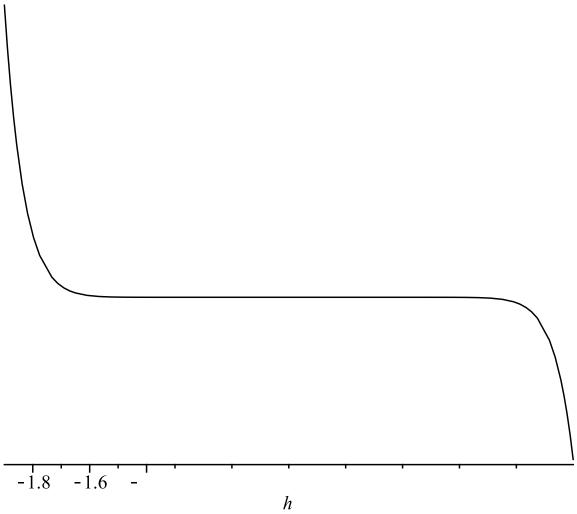

Fig. 1. $\hbar$-curve for the 15th order approximation ($c = 10$).

$$\phi(x;0) = u_0(x) \quad \text{and} \quad \phi(x;1) = u(x) \tag{11}$$

both hold. Therefore, as p increases from 0 to 1, the solution $\phi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x;p)$ in Taylor series with respect to p, one has

$$\phi(x;p) = \phi(x;0) + \sum_{m=1}^{+\infty} u_m(x)\, p^m, \tag{12}$$

where

$$u_m(x) = \frac{1}{m!} \left. \frac{\partial^m \phi(x;p)}{\partial p^m} \right|_{p=0}. \tag{13}$$

Now the convergence of the series (12) depends on the parameter $\hbar$. Assuming that $\hbar$ is chosen so properly that the series (12) is convergent at $p = 1$, we have, by means of (11), the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) \tag{14}$$

which must be one of the solutions of the original problem (1,2), as proved by Liao in [9].

The next goal is to obtain the higher order terms $u_m(x)$. Differentiating the zeroth order deformation equation (5) and its boundary conditions (10) m times with respect to p, then setting $p = 0$, finally dividing them by $m!$, we obtain the mth order deformation equation and its boundary conditions:

$$u_m^{(6)}(x) = \chi_m u_{m-1}^{(6)}(x) + \hbar R_m(\vec{u}_{m-1}(x)), \tag{15}$$

Table 1
Relative errors of HAM approximations (c = 10).

x	5th order	10th order	15th order
0.1	6.9E-11	3.0E-16	5.4E-22
0.2	2.6E-10	6.1E-16	8.0E-22
0.3	1.1E-9	8.7E-16	9.0E-22
0.4	1.7E-9	1.1E-15	9.2E-22
0.5	1.9E-9	1.1E-15	8.7E-22
0.6	1.5E-9	9.2E-16	7.8E-22
0.7	7.6E-10	6.3E-16	6.5E-22
0.8	1.6E-10	3.7E-16	4.9E-22
0.9	3.5E-11	1.5E-16	2.7E-22

$$u_m(0) = u'_m(0) = u''_m(0) = u_m(1) = u'_m(1) = u''_m(1) = 0; \qquad (16)$$

where

$$R_m(u_{m-1}(x)) = u$$

Fig. 2. $\hbar$-curve for the 15th order approximation (c = 1000).

$$
\begin{aligned}
u_1(x) = {} & \frac{\hbar c}{5040} x^{10} + \frac{\hbar c}{12096\,e}\left(7\,e^2 - 24\,e - 19\right) x^9 \\
& - \frac{\hbar}{3360\,e}\left(60\,e + 9\,c\,e^2 + 38\,c\,e - 23\,c\right) x^8 \\
& + \frac{\hbar}{3360\,e}\left(351\,c - 127\,c\,e^2 - 140\,e^2 + 480\,e + 452\,c\,e + 380\right) x^7 \\
& + \frac{\hbar}{60\,e}\left(38\,e - 22\,c\,e - 23\,c + 9\,c\,e^2 + 9\,e^2 - 23\right) x^6 \\
& - \frac{\hbar}{20160\,e}\left(4135\,c\,e^2 - 9643\,c - 8700\,c\,e + 4032\,e^2 + 51984\,e - 9504\right) x^5 \\
& + \frac{\hbar}{10080\,e}\left(1176\,e^2 - 2513\,c - 2472 + 27972\,e - 2360\,c\,e + 1223\,c\,e^2\right) x^4 \\
& + \frac{\hbar}{15120\,e}\left(666 - 378\,e^2 - 14436\,e + 723\,c\,e + 674\,c - 398\,c\,e^2\right) x^3.
\end{aligned} \tag{20}
$$

$u_m(x)(m = 2, 3, \ldots)$ can be calculated similarly.

The mth order approximation can be generally expressed as

$$u(x;\hbar) \approx \sum_{k=0}^{m} u_k(x)\text{Them}$$

Table 2
Relative errors of HAM approximations (c = 1000).

x	5th order	10th order	15th order
0.1	9.1E-6	9.7E-6	1.9E-6
0.2	1.6E-4	2.9E-5	1.7E-6
0.3	4.4E-4	5.5E-5	3.1E-7
0.4	6.8E-4	7.6E-5	1.2E-6
0.5	7.3E-4	8.0E-5	1.7E-6
0.6	5.8E-4	6.5E-5	1.0E-6
0.7	3.2E-4	4.0E-5	2.2E-7
0.8	9.8E-5	1.8E-5	1.1E-6
0.9	4.7E-6	5.0E-6	9.8E-7

Let $\xi \in [0,1]$. Then $u(\xi;\hbar)$ is a function of $\hbar$, and the curve $u(\xi;\hbar)$ versus $\hbar$ contains a horizontal line segment which corresponds to the valid region of $\hbar$. The reason is that all convergent series given by di®erent values of $\hbar$ converge to its exact value. So, if the solution is unique, then all of these series converge to the same value and therefore there exists a horizontal line segment in the curve. We call such kind of curve the *$\hbar$-curve*; see Figure 1 for example, where the valid region of $\hbar$ is about $-1.6 < \hbar < -0.2$.

Although the solution series given by di®erent values in the valid region of $\hbar$ converge to the same exact solution, the convergence rates of these solution series are usually di®erent. A more accurate solution series can be obtained by assigning $\hbar$ a proper value which usually can be obtained by observation.

Now we are in a position to show how the parameter c in the problem (1,2) a®ects the approximate solution (21), 13113ms6(apprte)-one(1311326(c),)-2l364(wa27(b)6ects)-get(

Fig. 3. $\hbar$-curve for the 15th order HAM approximation ($c = 10^8$).

by the formula

$$\delta(x) = \left| \frac{u_{exact}(x) - u(x; \hbar)}{u_{exact}(x)} \right|, \tag{22}$$

where $u_{exact}(x)$ is the exact solution (3), and $u(x; \hbar)$ is the approximate solution (21).

(II) *Large values of c.* In this case, we take $c = 1000$ as an example. As pointed out in [8], the Adomian's decomposition method is no longer valid for this case.

To find the valid region of $\hbar$, the $\hbar -$ curve given by the 15th order approximation (21) when $c = 1000$ and $x = \frac{1}{2}$ is drawn in Figure 2, which clearly indicates that the valid region of $\hbar$ is about $-0.13 < \hbar < -$

Table 3
Relative errors of HAM approximations($c = 10^8$).

x	5th order	10th order	15th order
0.1	4.9E-6	2.9E-5	1.4E-5
0.2	1.7E-4	7.1E-5	1.9E-5
0.3	5.3E-4	1.2E-4	1.9E-5
0.4	8.4E-4	1.5E-4	1.7E-5
0.5	9.1E-4	1.6E-4	1.5E-5
0.6	7.1E-4	1.3E-4	1.4E-5
0.7	3.8E-4	8.5E-5	1.4E-5
0.8	1.1E-4	4.3E-5	1.2E-5
0.9	2.5E-6	1.5E-5	7.3E-6

are

$$[c, \hbar] = \left(1, -\frac{49}{50}\right), \left(10, -\frac{23}{25}\right), \left(100, -\frac{57}{100}\right), \left(10^3, -\frac{59}{500}\right). \tag{23}$$

Then we use the rational interpolation technique to find a rational function in c that interpolates the given points (23), which gives a relationship between c and $\hbar$:

$$\hbar(c) = -\frac{342960750 + 1115829\, c}{347425000 + 3665200\, c + 8350\, c^2}. \tag{24}$$

Substituting (24) into the mth order approximation (21) gives a solution expression

$$u(x, \hbar(c)) \approx \sum_{k=0}^{m} u_k(x) = \sum_{k=0}^{4m+6} \gamma_{m,k}(c)\, x^k, \tag{25}$$

which only depends on the parameter c. It turns out that from (25) one can always get a convergent series solution which agrees very well with the exact solution (3), no matter what value of c is.

For over 1000 random values of c in the interval $[1, 10^{30}]$, we have calculated the relative errors of the 15th order approximation (25) at different points in the interval $(0, 1)$ as in the case (I), and found that all these relative errors are less than 5×10^{-5}. Figure 4 shows that the 15th order approximation (25)

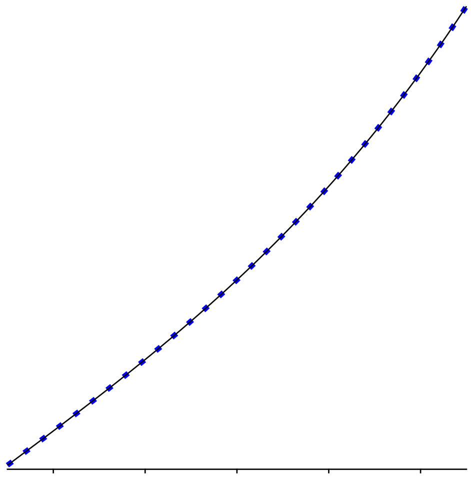

Fig. 4. Symbols: 15th order HAM approximation (21); solid line: exact solution (3).

where $s_{i,j}$ and $t_{i,j}$ are real numbers. Due to the continuity of $\hbar$ on c, equation (24) leads to good approximations for small values of c; while for large values of c, it is seen from (26) that

$$\lim_{c \to \infty} \varphi_{m,k}(c) = \frac{s_{2m,k}}{t_{2m,k}} \tag{27}$$

is independent of c. Therefore, equation (25) always give good approximation, no matter what value of c is.

3 Conclusions

In this paper, the homotopy analysis method (HAM) is successfully applied to solve a parameterized sixth order boundary value problem which, for large parameter values, cannot be solved by other analytical methods for finding approximate series solutions. The success mainly lies in the fact that the HAM provides a convergence-control parameter $\hbar$ which can be used to adjust and control the convergence region and rate of the series solution obtained, according to the value of the parameter. Therefore, the HAM is a promising analytical tool for solving nonlinear as well as linear problems.

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