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to all orders. In the appendix by Van Dyke, two dimensional problems which made the mistake, however, of writing an infinite sum where now we write a Henkel and $Q_d = 0$. An attempt to calculate $Q_b^{(1)}$ from (3.3b) by this method leads to an integrand $e^{-2s} \coth s$ which would give an infinite value for $Q_b^{(1)}$. Obviously the approximation $\phi^{(1)}$ breaks down in the neighbourhood of the gap where its gradient

4. The Solution in the Gap

[redacted]

To calculate the quantities defined in Section 2, we shall need $\partial\phi/\partial n$ on the sphere surface within the gap; this is most easily expressed in terms of Z:

[redacted]

$$= \varepsilon^{-1}/Z + \tfrac{1}{3}(1 + 1/Z) + O(\varepsilon). \tag{4.4}$$

[redacted]

result

$$\eta^2 \int_0^\infty f(s) J_0(s\eta)\, ds = \int_0^\infty [(f/s)' - f''] J_0(s\eta)\, ds, \tag{5.1}$$

which is true provided $f' - f/s \to 0$ as $s \to 0$. The prime denotes d/ds. For problem (c) the boundary condition becomes

$$\phi_c^{(2)}(1,\eta) = -(1+\eta^2)^{\frac{1}{2}} \int_0^\infty C(s) \sinh s J_0(s\eta)\, ds,$$

where

$$C(s)\sinh s = \mathrm{e}^{-s} - 3s\,\mathrm{e}^{-s} + 2s^2\,\mathrm{e}^{-s} + s^2\,\mathrm{e}^{-s}\coth s - (s\,\mathrm{e}^{-s}\coth s)' + (s^2\,\mathrm{e}^{-s}\coth s)''.$$

The required solution for $\phi_c^{(2)}$ is then obviously

$$\phi_c^{(2)}(\xi,\eta) = -(\xi^2+\eta^2)^{\frac{1}{2}} \int_0^\infty C(s) \sinh s\xi J_0(s\eta)\, ds.$$

For problem (b) things are not so straightforward. Before we can use (5.1) we must separate the s^{-1} singularity in $\coth s$; we obtain

$$\phi_b^{(2)}(1,\eta) = -\tfrac{1}{2}(1+\eta^2) + (1+\eta^2)^{\frac{1}{2}} \int_0^\infty B(s) \sinh s J_0(s\eta)\, ds, \tag{5.2}$$

where

$$B(s)\sinh s = s\,\mathrm{e}^{-s} + \tfrac{1}{2}s\,\mathrm{e}^{-s}\coth s - \tfrac{1}{2}(\mathrm{e}^{-s}\coth s - \mathrm{e}^{-s}/s)' + \tfrac{1}{2}(s\,\mathrm{e}^{-s}\coth s - \mathrm{e}^{-s})''.$$

We must now find a particular integral of Laplace's equation that will fit this boundary condition. We find it by substituting

[redacted]

into Laplace's equation, taking an inverse Hankel transform and obtaining an equation for W:

$$\partial^2 W/\partial\xi^2 - s^2 W + 4\int_0^\infty \xi(\xi^2+\eta^2)^{-\frac{1}{2}} \eta J_0(s\eta)\, d\eta = 0.$$

[redacted]

We now proceed to calculate Q, $\mathbf{S}$ and $\mathbf{F}$ and compare the expressions with other work.

6. Calculation of Q

[redacted]

$Q_c^{(2)}$, which comes easily from $\phi_c^{(2)}$. We must first express our definition (2.1) in terms of the (ξ, η, θ) coordinates we have been using. At first sight it may seem that this requires expressions for $\partial\phi/\partial n$ and dA on $\xi = 1 + \frac{1}{2}\varepsilon(\eta^2 - 1)$, but Q is independent of the surface chosen, provided it surrounds only one sphere, and so we choose the surface $\xi = 1$. Then

$$Q = \int \left[\frac{\partial \phi}{\partial \xi}\right]_{\xi=1} \frac{2\eta\, d\eta}{1+\eta^2}.$$

The calculation of $Q_c^{(2)}$ follows exactly that of $Q_c^{(1)}$, giving

[redacted]

where all integrals can be evaluated using integration by parts and Gradshteyn & Ryzhik (1965, Section 3.552).

[redacted]

[redacted]

$$Q_{bo} = T_b \ln Z_0 + \varepsilon \tfrac{1}{3} T_b (\ln Z_0 + Z_0 + 1) + O(\varepsilon^2).$$

Substituting for Z_0 gives

$$Q_{bo} = T_b[\ln 2 - \ln \varepsilon - \ln (1+\eta_0^2)] +$$

[redacted]

Adding Q_{bi} and Q_{bo} cancels the singular terms and gives

[redacted]

We now compare this result with those derived by Keller (1963) and Batchelor &

[redacted]

which, as stated, contains no singular terms. Thus, combining this with the fact that only the z component of $\mathbf{S}$ is non-zero, we can reduce our expression for S to

$$S = 2\pi a^2 \int_0^\infty \left\{ \left[\frac{\partial \phi}{\partial \xi} \right]_{\xi=1} - \varepsilon(1-z)\phi^{(1)} \right\} \frac{4\eta \, d\eta}{(1+\eta^2)^2} + O(\varepsilon^2).$$

Performing the integrations gives (using ζ for the Riemann zeta function)

$$S_{bc} = \tfrac{4}{3}\pi a^2 T_{bc}[\tfrac{1}{2}\pi^2 + \varepsilon(\tfrac{1}{3}\pi^2 - \tfrac{1}{2})] - \tfrac{4}{3}\pi a^2 G_c[6\zeta(3) + \varepsilon(4\zeta(3) - 3\zeta(4))], \qquad (7.1)$$

where T_{bc} is given by (6.1), and $S_d^{(1)} = 3\pi a^2 \zeta(3) G_d$. Smith & Rungis (1975) and Love (1975) obtained the leading-order G_c term for S_{bc} but not the T_{bc} term.

The expressions with which we shall compare these results come from Jeffrey (1973) and are exact for all separations; in the present notation,

$$S_{bc} = \tfrac{4}{3}\pi a^2 G_c \sum_{p=0}^{\infty} A_{01p} t^p \quad \text{and} \quad S_d = \tfrac{4}{3}\pi a^2 G_d \sum_{p=0}^{\infty} A_{11p} t^p,$$

where $t = \tfrac{1}{2}(1+\varepsilon)^{-1}$ and the coefficients are given by

$$A_{mn0} = 3\delta_{1n} \quad \text{and} \quad A_{mnp} = (-1)^m \sum_{s=1}^{(p-n-3)/2} \binom{n+s}{n+m} A_{ms(p-n-s-1)}.$$

We wish to test the rate at which the infinite series converge, both to test their

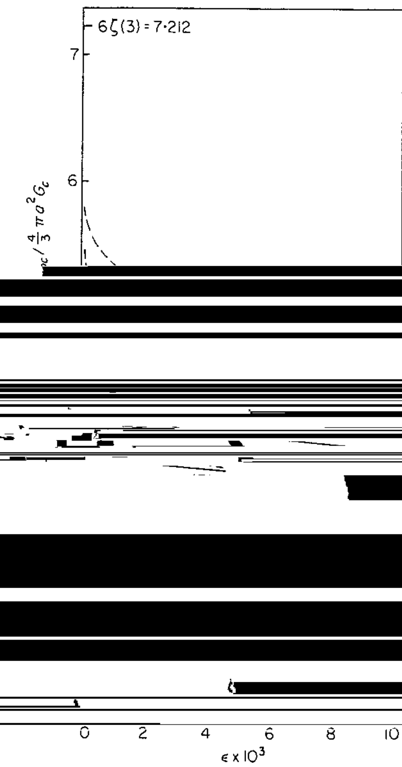

FIG. 2. Comparison of different expressions for S_{bc}. The broken lines actually start from 7·212 at $\varepsilon = 0$. The curves are: (1) $6\zeta(3)+\frac{1}{6}\pi^4/\ln\varepsilon$; (2) $6\zeta(3)+\frac{1}{6}\pi^4/(\ln\varepsilon-\ln 2-2\gamma)$; (3) the series summed to $P = 70$; (4) the series summed to $P = 150$.

F_{bc}, which has not yet been studied. Warren & Cuthrell (1975) have calculated F_b and also measured it experimentally; Davis (1964) has calculated F_{bc}; Smith & Barakat (1975) have calculated F_c. Since $T_{bc} \to 0$ as $\varepsilon \to 0$, one might expect that $F_{bc} \to F_c$ as $\varepsilon \to 0$. The numerical results of Davis (1964) and Smith & Barakat (1975) throw doubt on this and here we shall see explicitly that it is not true. In fact $F_{bc} \to \infty$ as $\varepsilon \to 0$.

Only the z components of $\mathbb{F}_b$, $\mathbb{F}_c$ and $\mathbb{F}_{bc}$ are non-zero (neglecting possible complications produced by problem (d)). Squaring (4.4) and integrating appropriately gives the contribution from the gap as

$$F_{bo} = 2\pi T_b^2\{\varepsilon^{-1}(1-1/Z_0)-\tfrac{1}{3}\ln Z_0+\tfrac{5}{3}(1-1/Z_0)\}+O(\varepsilon)$$

$$= 2\pi T_b^2\{\varepsilon^{-1}+\tfrac{1}{3}\ln\varepsilon+\tfrac{5}{3}-\tfrac{1}{3}\ln 2-\tfrac{1}{2}(1+\eta_0^2)+\tfrac{1}{3}\ln(1+\eta_0^2)\}+ O(\varepsilon)+O(\eta_0^{-2}).$$

Note that both terms of the gap solution are needed to cancel the singularities in the first term of the solution outside the gap. The contribution of (3.3b) to the force is

$$F_{bi}^{(1)} = 2\pi T_b^2 \int_0^{\eta_0}\left[\frac{\partial\phi_b^{(1)}}{\partial\xi}\right]_{\xi=1}^2 \frac{\eta^2-1}{\eta^2+1}\,\eta\,d\eta. \tag{8.1}$$

We first separate those terms in $\partial\phi/\partial\xi$ which will lead to singular terms in η_0. Thus

$$\left[\frac{\partial\phi_b^{(1)}}{\partial\xi}\right]_{\xi=1} = 1+\frac{2/3}{1+\eta^2}+(1+\eta^2)^{\frac{1}{2}}\int_0^\infty s\,e^{-s}\left(\coth s-\frac{1}{S}+\tfrac{1}{3}\right)J_0(s\eta)\,ds.$$

Squaring this and integrating according to (8.1) gives

It will be seen below that the singular terms alone in F_b will be sufficient to reproduce [redacted]

$$F_b^{(1)} = 2\pi T_b^2(\varepsilon^{-1} + \tfrac{1}{3}\ln\varepsilon + O(1)). \qquad (8.2)$$

[redacted]

Appendix: Two Almost-Touching Circles

M. VAN DYKE

If one replaces the spheres of the main paper by parallel, almost touching circular

cylinders, the various approximate solutions to problems (b)–(d) can be found in simple closed form. There is no counterpart of problem (a), for if two circles are at equal temperatures there is a logarithmic singularity in the potential at infinity unless the temperature is constant everywhere.

[illegible] (A.6)

[redacted]

where $H = 1+\frac{1}{2}Y^2$ as one would expect. The first term is the solution of Keller (1963). Although we have made no use of the boundary condition far from the circles, this matches with the solution there (i.e. this outer approximation matches the inner approximation (A.1)). That is, rewriting the one-term solution (A.1) in the gap variables X, Y, expanding for small ε and keeping two terms gives $2X/Y^2 - 2\varepsilon X^3/Y^4$, conversely, rewriting the two-term solution (A.6) in the variables used outside the gap x, y, expanding for small ε and keeping one term gives $2x/y^2 - 2x^3/y^4$; and these are the same.

This might suggest that in this singular perturbation problem matching has played no role at this stage; but that is not true. The solution (A.1) is in fact not unique, for [illegible]

[redacted]

$$\sin N\pi\xi \cosh N\pi\eta, \quad N = 1, 2, 3 \ldots \tag{A.7}$$

[illegible]

[redacted]

periodic array of cylinders). Thus the singularities in the last term of (A.10) must also cancel. To verify this, we continue the solution outside the gap in the form $\phi^{(1)}+\varepsilon\phi^{(2)}+\ldots$.

[redacted]

$$\phi_b^{(2)} = \tfrac{1}{6}\,\mathrm{Re}\,(2\zeta+\zeta^3) = \tfrac{1}{3}\xi+\tfrac{1}{6}(\xi^3-3\xi\eta^2). \qquad \text{(A.11)}$$

This shows a typical symptom of non-uniformity, being singular like the inverse cube

[redacted]