Recursive integration of piecewise-continuous functions

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Abstract

An algorithm is given for the integration of a class of piecewise-continuous functions. The integration is with respect to a real variable, because the functions considered do not in general allow integration in the complex plane to be defined. The class of integrands includes commonly occurring waveforms, such as square waves, triangular waves, and the floor function; it also includes the signum function. The algorithm can be implemented recursively, and it has the property of ensuring that integrals are continuous on domains of maximum extent.

1 Introduction

The integration of a function expressed using the Maple function `piecewise` or the signum function was considered in [3], where the fundamental definitions and theorems on integrating discontinuous functions were presented. We recall that a function $F(x)$ is said to have breakpoints at those values of x where the function is discontinuous. It was also pointed out in [3] that the problem of integrating piecewise-continuous functions can be posed only within the context of integration with respect to a real variable, and thus we continue to work in that context. The integration problem has two aspects: deriving a primitive, or anti-derivative for a given function, and ensuring that the result returned by the integrator is valid on a domain of maximum extent [2].

We also follow [3] in noting that a discussion of the integration of piecewise-continuous functions can be distracted by contentious, but irrelevant, issues, such as which definitions of the signum and Heaviside functions are the correct ones. These issues cannot be ignored completely, because the value of sgn(0) has a bearing on the results given here. However, we shall avoid the distraction by defining a cognate of the signum function that fulfills the requirements of integration, and use it without prejudice to the wider discussion.

The new features of the present algorithm are, first, an extension to a broader class of integrands. Specifically, it can

3 Preliminary transformations

This section establishes that an integrand containing any of the above functions can be reduced to one containing only floor and signum. These transformations are not in general equalities, because of pointwise differences, but they leave any integral unchanged.

$$
\begin{aligned}
|x| &\Rightarrow xS(x) \; . \\
x \bmod y &\Rightarrow x - y\lfloor x/y \rfloor \; . \\
H(x) &\Rightarrow \tfrac{1}{2}(1+S(x)) \; . \\
SQW(x) &\Rightarrow \cos(\pi\lfloor x \rfloor) \; . \\
STW(x) &\Rightarrow 2x - 1 - 2\lfloor x \rfloor \; .
\end{aligned}
$$

In addition, if a signum function has a nonlinear argument that can be factored, then we have the transformation

$$S((ax-b)g(x)) \Rightarrow S(ax-b)S(g(x)) \; .$$

Finally, the argument of S can be simplified by using

$$S(ax-b) \Rightarrow S(a)S(x-b/a) \; ,$$

where it is assumed that $a \neq 0$.

Therefore, any integration problem $\int f(x,\{P_i\})dx$ can be replaced by the problem

$$\int F(x,\{S(x-b_i)|i=1..p\},\{\lfloor c_j x - d_j \rfloor | j=1..q\})dx \; , \quad (6)$$

where for nontriviality it is assumed that $c_j \neq 0$.

4 Integration procedure

The integration problem (6) is solved in a way that can be programmed recursively. Integration starts by replacing all of the signum and floor functions by temporary symbolic constants; here symbols s_i are used for S functions and f_j for floor:

$$F \Rightarrow F(x,\{s_i|i=1..p\},\{f_j|j=1..q\})\; dx \; .$$

The standard procedures of the particular system can now be used to integrate $F(x,\{s_i\},\{f_j\})$ with respect to x to obtain the primitive $G(x,\{s_i\},\{f_j\})$.

Theorem: Let $G(x,\{s_i\},\{f_j\})$ be a primitive of the function $F(x,\{s_i\},\{f_j\})$, and let G be continuous in all its arguments. Then $G(x,\{S(x-b_i)\},\{\lfloor c_j x - d_j \rfloor\})$ is a primitive of $F(x,\{S(x-b_i)\},\{\lfloor c_j x - d_j \rfloor\})$.

Proof: The set of breakpoints of F is

$$\mathbb{D} = \{b_i | i=1..p\} \cup \{(n+d_j)/c_j | j=1..q, n \in \mathbb{Z}\}.$$

For any $x \notin \mathbb{D}$, there exists a neighbourhood of x within which the signum and floor functions are constant. Hence $G' = F$ within that neighbourhood by construction. Thus $G' = F$ for all $x \notin \mathbb{D}$. □

Although the function $G(x,\{s_i\},\{f_j\})$ is assumed to be continuous, after the temporary constants are returned to their signum and floor functions, there will be discontinuities at the breakpoints in general. To obtain an integral valid on the domain of maximum extent [2], these discontinuities must be removed. The function G can be regarded as a candidate for the integral of F, but one that must be rectified, i.e. made continuous. The algorithm continues by choosing one of the temporary constants and substituting the original function for it, and then eliminating any discontinuities thereby introduced before continuing to the next constant.

Starting with the first signum, we proceed as follows.

$$G(x,\{s_i\},\{f_j\}) \Rightarrow G(x,S(x-b_1),\{s_i|i=2..p\},\{f_j\}).$$

Now compute

$$
\begin{aligned}
J &= G(b_1,1,\{s_i|i=2..p\},\{f_j\}) \\
&\quad - G(b_1,-1,\{s_i|i=2..p\},\{f_j\}) \; ,
\end{aligned}
$$

and define

$$
\begin{aligned}
&G_1(x,\{s_i|i=2..p\},\{f_j\}) = \\
&G(x,S(x-b_1),\{s_i|i=2..p\},\{f_j\}) - \tfrac{1}{2}JS(x-b_1) \; .
\end{aligned}
$$

Theorem: The function G_1 is a primitive of F and is continuous at $x = b_1$.

Proof: If $x \neq b_1$, then $S(x-b_1)$ is a constant and therefore $G_1' = G'$. By direct computation

$$
\begin{aligned}
\lim_{x\to b_1+} G_1 = \lim_{x\to b_1-} G_1 &= \tfrac{1}{2}G(b_1,1,\{s_i|i=2..p\},\{f_j\}) \\
&+ \tfrac{1}{2}G(b_1,-1,\{s_i|i=2..p\},\{f_j\}) \; ,
\end{aligned}
$$

where the continuity of G has been used. □

The function G is now discarded and G_1 used instead. Thus G_1 is the new candidate for the integral. As each s_i is returned to $S(x-b_i)$, a new function G_i is created and used in subsequent steps. After p steps, all the constants $\{s_i\}$ have been returned to their signum functions and a function G_p has been computed that is continuous at all breakpoints, and is the candidate for an integral.

A more elaborate procedure is needed in order to return the floor functions to the candidate function. Again the first constant is replaced:

$$G_p(x,\{f_j\}) \Rightarrow G_p(x,\lfloor c_1 x - d_1 \rfloor,\{f_j|j=2..q\}) \; .$$

The floor function is discontinuous when its argument equals an integer, when $c_1 x - d_1 = n$. Therefore calculate the jump

$$
\begin{aligned}
J_n &= G_p\left(\frac{n+d_1}{c_1}, n, \{f_j|j=2..q\}\right) \\
&\quad - G_p\left(\frac{n+d_1}{c_1}, n-1, \{f_j|j=2..q\}\right) \; ,
\end{aligned}
$$

and define a new candidate function by

$$
\begin{aligned}
G_{p+1}(x,\{f_j|j=2..q\}) &= G_p(x,\lfloor c_1 x - d_1 \rfloor,\{f_j|j=2..q\}) \\
&\quad - \sum_{m=1}^{\lfloor c_1 x - d_1 \rfloor} J_m \; . \qquad (7)
\end{aligned}
$$

If $\lfloor c_1 x - d_1 \rfloor < 0$, the summation in this formula is evaluated using the convention of Graham, Knuth and Patashnik [1], given in the answer to their exercise 2.1, to wit, if $k < j$, then

$$\sum_{m=j}^{k} P(m) = -\sum_{m=k+1}^{j-1} P(m) \; . \qquad (8)$$

Notice that for $k = j - 1$, corresponding in the definition of G_{p+1} to $\lfloor c_1 x - d_1$

Using the methods given here, its integral is given by

$$\int HW(x)\,dx = \frac{1}{\pi}(2\lfloor x\rfloor - \cos\pi x(1+\cos\pi\lfloor x\rfloor))\ . \qquad (12)$$

Similarly the full-wave rectified sinewave is studied. It is defined to be $FW(x) = SQW(x)\sin\pi x$. However it can be equivalently expressed as $\sqrt{1-\cos 2\pi x}/\sqrt{2}$, in which form it was studied in [4], where an integral was obtained that is equivalent to that obtained by the present method.

$$\int FW(x)\,dx = \frac{2\lfloor x\rfloor}{\pi} - \frac{\cos\pi x\cos\pi\lfloor x\rfloor}{\pi}\ . \qquad (13)$$

In [4], continuous integrals were obtained for trigonometric functions by including floor functions in the final results. Could there be a difficulty if an integrand studied in [4

This paper has focussed on the integration of piecewise-continuous functions. We do not wish to imply by this that the algebra of these functions could not benefit from further work. Rather the opp