Integration of the signum, piecewise and related functions

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Abstract

When a computer algebra system has an assumption facility, it is possible to distinguish between integration problems with respect to a real variable, and those with respect to a complex variable. Here, a class of integration problems is defined in which the integrand consists of compositions of continuous functions and signum functions, and integration is with respect to a real variable. Algorithms are given for

The interest in this class of problems arises because functions that have piecewise definitions are widely used in engineering, physics, and other areas. Such functions are often constructed explicitly by users of CAS to represent discontinuous processes. They can also appear as the result of algebraic simplifications performed by a CAS on an integrand, even if that integrand contained no signum functions explicitly when first presented. An important feature of the computations discussed here is the fact that they ensure that the expressions obtained are valid on domains of maximum extent.

We remark that functions equivalent to signum are supported by all the major CAS. However, the support takes various forms and the definitions used by the different systems are not completely equivalent. Examples include the SIGN function in Derive, the `signum` and `piecewise` functions in Maple V and the `UnitStep` in Mathematica.

2 Definitions of functions

The signum function is defined differently in each of the major CAS. This is not really surprising given that different areas of mathematics also use different definitions of a signum function. However, these disagreements do not affect the integration question, and a discussion of variations would only distract attention from the main problem. Therefore one particular definition, and a specific unambiguous notation, is used here, so that the issue of variations in definition does not intrude on this discussion of integration. A signum function $S_{nn} : \mathbb{R} \to \mathbb{R}$ that is 1 for all non-negative real numbers, briefly an n-n signum, is defined by

$$S_{nn}(x) = \begin{cases} 1\ , & \text{for } x \geq 0, \\ -1\ , & \text{for } x < 0. \end{cases} \tag{3}$$

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A similar calculation shows the left limit has the same value. Finally, substituting $x = x^{\rm b}$ into the right-hand side of (14) gives

$$\begin{aligned} G(x^{\rm b}) &= g(x^{\rm b}, S_{nn}(0)) - JS_{nn}(0) = g(x^{\rm b}, 1) - J \\ &= \tfrac{1}{2}g(x^{\rm b}, 1) + \tfrac{1}{2}g(x^{\rm b}, -1) \ . \end{aligned}$$

Therefore, G is continuous as required. Notice that for many other signum functions, in particular for the one defining $\mathrm{sgn}(0) = 0$, the theorem would not be true. □

We now generalize the theorem to integrands containing a finite number of signum functions.

Theorem 5: Let $f(x, \{s_i | i = 1..p\})$ be a function that is continuous with respect to x on $[a, b]$, and let $\{x_i^{\rm b} | i = 1..p\}$ be a set of distinct points with $x_1^{\rm b} > x_2^{\rm b} > \ldots > x_p^{\rm b}$. Let $g(x, \{s_i | i = 1..p\})$ be an integral of f on $[a, b]$, then

$$\begin{aligned} &\int f(x, S_{nn}(x - x_i^{\rm b}) | i = 1..p\}) dx \\ &\quad = G(x) = g(x, \{S_{nn}(x - x_i^{\rm b}) | i = 1..p\}) \\ &\qquad - \sum_{i=1}^{p} J_i S_{nn}(x - x_i^{\rm b}) \ , \end{aligned} \tag{15}$$

where

$$\begin{aligned} J_j &= \tfrac{1}{2} g(x_j^{\rm b}, S_j^{<}, 1, S_j^{>}) - \tfrac{1}{2} g(x_j^{\rm b}, S_j^{<}, -1, S_j^{>}) \ , \\ S_j^{<} &= \{\mathrm{sgn}(x_j^{\rm b} - x_i^{\rm b}) = -1 | i < j\} \ , \\ S_j^{>} &= \{\mathrm{sgn}(x_j^{\rm b} - x_i^{\rm b}) = 1 | i > j\} \ . \end{aligned}$$

Proof. Let the interval $[a, b]$ be partitioned into subintervals $[a_i, a_{i+1}]$, where $a_0 = a$, $x_i^{\rm b} < a_i < x_{i+1}^{\rm b}$ and $a_{p+1} = b$. By the previous theorem, (15) is an integral on each $[a_i, a_{i+1}]$, and by construction and hypothesis it is continuous on each $[x_i^{\rm b}, x_{i+1}^{\rm b}]$, therefore it is continuous and an integral on $[a, b]$. □

In the case of integration on an unspecified domain we require.

Theorem 6. If $f(x, \{s_i | i = 1..p\})$ is integrable with respect to x, for $\{s_i\}$ fixed and equal to ± 1, on domains $R_i \subset R$, and $g(x, \{s_i\})$ is the integral of f on the domain of maximum extent, i.e. $\bigcup R_i$, then G(x) defined in (15) is also an integral on the domain of maximum extent, provided $J_i = 0$ if $x_i^{\rm b} \notin \bigcup R_i$. □

The next theorem applies to the case in which an integrable singularity coincides with the breakpoint of a signum function.

Theorem 7. Let $x \in [a, b]$ and $s = \pm 1$, and let $f(x, s)$ be a function such that f is continuous with respect to x on $[a, b]$ except at $x_b \in [a, b]$. Let $f(x, s)$ be integrable at $x^{\rm b}$. Then

$$\int f(x, S_{nn}(x - x^{\rm b})) dx = G(x) = g(x, S_{nn}(x - x^{\rm b})) \ , \tag{16}$$

where $g(x, s)$ is an integral of $f(x, s)$ on $[a, b]$ subject to $g(x^{\rm b}, s) = 0$.

Proof. Let $g_a(x, s) = \int_a^x f(y, s) dy$. By theorem 3, g_a is continuous on $[a, b]$. Define $g(x, s) = g_a(x, s) - g_a(x^{\rm b}, s)$. For $x < x^{\rm b}$,

$$\partial_x g(x, S_{nn}(x - x^{\rm b})) = \partial_x g(x, -1) = f(x, S_{nn}(x - x^{\rm b})) \ ,$$

and likewise for $x > x^{\rm b}$. At $x^{\rm b}$,

$$\lim_{x \to x^{\rm b}-} G(x) = \lim_{x \to x^{\rm b}+} = G(x^{\rm b}) = 0$$

□

5 Integration of signum

The above theorems can be summarized in the following algorithm. Given an integral with respect to a real variable containing signum or Heaviside functions, the algorithm used by Derive roughly proceeds as follows.

1. Use the definitions (4) to convert Heaviside to signum functions.
2. Check each signum has a linear argument, and replace each $\mathrm{sgn}(\alpha_i x + \beta_i)$ with $S_{nn}(\alpha_i) S_{nn}(x - x_i^{\rm b})$. If any signums contain other arguments, the algorithm fails.
3. Order the breakpoints so that the integrand is in the form $\hat{f}(x, \{S_{nn}(x - x_k^{\rm b}) | k = 1..p\})$ where p is an integer, the $\{x_k^{\rm b}\}$ are the ordered breakpoints of the integrand, with $x_1^{\rm b} < x_2^{\rm b} < \ldots < x_p^{\rm b}$. Further the function $\hat{f}(x, \{s_k | k = 1..p\})$, with the $\{s_k\}$ being symbolic constants, contains no signum function.
4. Pass the function $\hat{f}(x, \{s_k\})$ to the system integrator. Assume it returns a function $g(x, \{s_k\})$, else FAIL.
5. For k from 1 to p, compute

   $$J_k = \tfrac{1}{2} G(x_k^{\rm b}, \{S^{<}, 1, S^{>}\}) - \tfrac{1}{2} G(x_k^{\rm b}, \{S^{<}, -1, S^{>}\})$$

   where $S^{<}$ is a set of $k - 1$ entries equal to 1 and $S^{>}$ is a set of $p - k$ entries -1.
6. Return the [illegible] the [illegible] $x - x$ entri

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we find $J = 2$ and hence obtain

$$\int 3x^2\sqrt{1+1/x^2}\,\mathrm{d}x = S_{nn}(x)\left[(1+x^2)^{3/2}-1\right] .$$

Continuity at the origin has already been noted. □

Example 2. In this example we illustrate the need for definition (3). For the integral of $(x+2)^{1+\operatorname{sgn} x}$, $f(x,s) = (x+2)^{1+s}$ and $g = (x+2)^{2+s}/(2+s)$ and $J = 2/3$. Thus

$$\int (x+2)^{1+\operatorname{sgn} x} dx = \frac{(x+2)^{2+S_{nn}(x)}}{2+S_{nn}(x)} - \frac{S_{nn}(x)}{3} \tag{17}$$

At $x = 0$, this evaluates to the correct $5/3$ using definition (1). In contrast, the other common definition in which $\operatorname{sgn}(0) = 0$ would yield the value 2 and hence create a removable discontinuity at $x = 0$. □

Example 3. The algorithm relies on the underlying integration system to return a continuous expression for $g(x,s)$, in the notation of the theorems. For example, the result

$$\int \frac{3\operatorname{sgn}(x-\pi)}{5-4\cos x} dx = \left(x - \pi + 2\arctan\frac{\sin x}{2-\cos x}\right) S_{nn}(x-\pi)$$

cannot be obtained if the system computes the integration: $\int 3s/(5-4\cos x)dx = 2s\arctan(3\tan(x/2))$. □

Example 4. Note that there is no difficulty if the integrand is singular at the break point of a signum. For example,

$$\int \frac{\operatorname{sgn} x\, \mathrm{d}x}{x^{1/3}} = \tfrac{3}{2}x^{2/3}S_{nn}(x) ,$$

where the fractional powers are interpreted as real-valued. □

Example 5. In this example we use Heaviside functions set in the context of a simple differential equation, even though the present algorithm applies only to integration. From engineering beam theory, the bending moment $M(x)$ in a beam that extends from $x = 0$ to $x = l$ and supports point loads P_a and P_b at $x = a$ and $x = b$ is given by the equation

$$\frac{dM}{dx} = \begin{cases} K, & \text{for } 0 \le x \le a ; \\ K + P_a, & \text{for } a \le x \le b ; \\ K + P_a + P_b, & \text{for } b \le x \le l . \end{cases}$$

Here K is a constant to be determined from the boundary conditions, which are $M(0) = M(l) = 0$ for the case of free ends. Engineers commonly solve equations like this using a specialised system of notation called Macaulay brackets [2], which essentially develop a subset of the results above. Instead, the equation is written

$$\frac{dM}{dx} = K + P_aH(x-a) + P_bH(x-b) , \tag{18}$$

and integrated. The integral of the Heaviside function is

$$\int H(x-a)\,dx = \int (\tfrac{1}{2} + {}^{1}$$

which in turn reduces to

$$f(x)H(x-a) - f(x)H(x-b).$$

Lemma 9. Let $f : \mathbb{R} \to \mathbb{R}$ be specified by

$$f(x) = f_0(x) + \sum_{i=1}^{p} f_i(x)H(x - x^{\mathrm{b}}$$

$$1/2\,\frac{|2-|x||\,x\,(-4+|x|)}{-2+|x|}$$

leaving a result with discontinuities at -2 and 2. By converting first to piecewise functions we get a continuous integral of the function.

```
> f := convert(abs(2-abs(x)),piecewise);
```

$$f := \begin{cases} -2-x & x \le -2 \\ 2+x & x \le 0 \\ 2-x & x \le 2 \\ x-2 & 2 < x \end{cases}$$

```
> g := int(f,x);
```

$$g := \begin{cases} -2\,x-1/2\,x^2 & x \le -2 \\ 2\,x+4+1/2\,x^2 & x \le 0 \\ 2\,x+4-1/2\,x^2 & x \le 2 \\ -2\,x+8+1/2\,x^2 & 2 < x \end{cases}$$

```
> convert( g, abs );
```

$$-\frac{x}{2}\,|x|+(\frac{x}{2}+1)\,|x+2|+(\frac{x}{2}-1)\,|x-2|-2\,x+4$$

Some further applications can be found in [4]. □

7 Conclusions

The existing implementations in Derive and Maple are not completely reflected in this presentation. For example, the definition $S_{nn}(x)$ is not used by Derive. In Derive, Example 1 is evaluated at $x = 0$ by taking a limit. The signum function in Maple can be modified to make it act like $S_{nn}(x)$ by setting an environment variable.

Although similar facilities may be present in other systems, we do not have access to them.

The correct integration of piecewise-continuous functions is not solely a CAS issue. The average user of a CAS has received little instruction from elementary mathematics books on working with functions as simple as $|x|$ — indeed no table of integrals contains an entry for this function — and without that background users might be slow to accept them. In addition, the integration of piecewise functions requires users to understand the difference between integration with respect to a complex variable and with respect to a real variable. There has already been a significant impact by CAS on the practice and teaching of mathematics, and piecewise-continuous functions could be another area in which CAS will lead the way.

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