

The Solution of $S \exp(S$

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ABSTRACT

We study the solutions of the matrix equation $\exp(\tau) = A$. Our motivation comes from the study of systems of delay differential equations

ferentiable function (τ) such that

$$(\tau) = A (\tau - 1)$$

where A is an n -by- n matrix of complex numbers, and (τ) is specified on an initial vector history

$$(\tau) = (\tau) = [\tau_1(\tau) \tau_2(\tau) \dots \tau_n(\tau)]^T$$

on the interval $-1 \leq \tau \leq 0$.

As in [7], this is a special problem, useful for some models in mathematical biology and elsewhere. Powerful numerical techniques exist for solving general delay differential equations; see for example [14]. The ultimate aim of the present work, in contrast, is to look for special-purpose techniques that may be more efficient for these 'niche' problems, or give greater insight. The approach used here is to note that the ansatz

$$(\tau) = \exp(\tau \mathbf{c})$$

for some constant n -vector \mathbf{c} (later to be used as one term in a Fourier-like series solution of the delay differential equation) leads to some interesting matrix computations, such as the computation of any and all \mathbf{c} such that¹

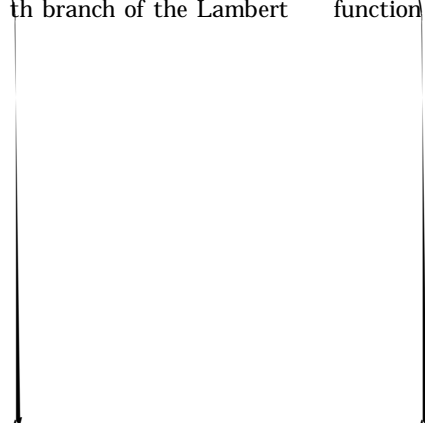
$$\exp(\mathbf{c}) = A \tag{1.1}$$

We shall consider this equation for $A \in \mathbb{C}^{n \times n}$.

Equation (1.1) is a matrix analogue of the scalar equation $s^s = A$, $s \in \mathbb{C}$, whose solutions are $s = \kappa_k(A)$, where κ_k is the k th branch of the Lambert W function [3]. Ma-

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the lessons learned in computing the matrix logarithm will be of use in solving (1.1) [2]. Here, we will barely scratch the surface of numerical computation for this problem, concentrating mainly on the theoretical aspects, and we will return to the problem of numerical computation in a future paper.

2. MATRIX FUNCTIONS AND EQUATIONS

We first recall some definitions. For a more comprehensive discussion, see [9].

Let $\mathcal{C} = \{z \in \mathbb{C} : |z| < r\}$. If

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

is a convergent power series in \mathcal{C} , then the

matrix function $f : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ is defined to be

$$f(A) = \sum_{k=0}^{\infty} c_k A^k \quad (2.1)$$

which converges for $\rho(A) < r$, where ρ is the spectral radius.

Recalling that any square matrix A has the Schur decomposition $A = T^{-1} D T$, where T is unitary and the diagonal entries of D

case the delay differential equation reduces to the ordinary differential equation $\dot{x}(t) = 0$ and is of little further interest. Some interest remains in this case when $\tau = 1$, however, as we shall see.

4. SOME GENERAL RESULTS

We are concerned from now on to discover the relations between the matrix function $\exp_k(A)$ defined above, and the solutions of equation (1.1). As with the functional definition (2.1), we can limit our discussion to upper triangular matrices A .

LEMMA 1. If $\exp_k(A) = B$, then A is upper triangular.

Proof. If $\exp_k(A) = B$, and $A = P^{-1}AP$ where P is upper triangular, then $P^{-1} \exp_k(A) P = \exp_k(P^{-1}AP)$ and if we put $C = P^{-1}A$ we see that $\exp_k(C) = P^{-1}B$, and thus any solution $x(t)$ of the original equation is similar to a solution of the same equation where the input $f(t)$ is upper triangular. Conversely, any solution $y(t)$ of the upper triangular equation gives a solution $x(t) = P y(t)$ of the original equation. \square

Henceforth we assume that A is upper triangular. One question that we can then ask is: if $\exp_k(A) = B$ and B is upper triangular must A also be upper triangular? We first establish another lemma.

LEMMA 2. If $\exp_k(A) = B$ and B is upper triangular, then A is upper triangular.

Proof

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5. THE T O-B -T O CASE

We now specialize to the case $n = 2$ in order to glean some more insight.

We consider first the case where A is derogatory, which for $n = 2$ implies $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Theorem 2 shows that there are infinitely many diagonalizable solutions to (1.1) for $\lambda \neq 0$. But these are not the only solutions. For example, if $A = \text{diag}(-1, -1)$ then an easy computation shows that the non-diagonalizable matrix

$$J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

satisfies $J^2 = -I$

Finally, we note that if X is not constrained to be upper triangular, then the problem of taking the matrix exponential (when the entries of X are symbols) as part of any procedure for solving $\exp(X) = A$ becomes rather complicated. Consider the 5×5 example at the end of Section 4; even with such a modest example, the nonlinear equations that arise in $\exp(X) = A$ are daunting for hand calculation, and the complexity grows rapidly enough with dimension that symbolic methods are in all likelihood not going to be useful for dimensions much larger than 5.

Special-purpose numerical schemes for take90.00-415.3(5)9.4(.)]25.1(o)-25.1(d)]TJ T* 0.0TD 0 Tc ak T* (the)aD 0 Tc 18.9(u)7.2(27o)-2(n)1

points; continua of solutions in the derogatory case; and vanishingly small impact of the roots of $\mu \exp \mu = \lambda$ when the eigenvalues λ of A are small.²

These difficulties should be borne in mind when inves-