Reducing expression size using rule-based integration

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Abstract. This paper describes continuing progress on the development of a repository of transformation rules relevant to inde nite integration. The methodology, however, is not restricted to integration. Several optimization goals are being pursued, including achieving the best form for the output, reducing the size of the repository while retaining its scope, and minimizing the number of steps required for the evaluation process. New optimizations for expression size are presented.

1 Introduction

The methods of integration can be conveniently divided into several categories.

- Look-up tables. These are collections or databases, such as [4], which try to list all possible integrals, each in a general form. Many special cases are also listed separately.
- Rule-based rewriting. The databases used are smaller than those for the look-up tables. They contain rules for transforming a given integral into one or more simpler integrals, together with rules for completing the evaluation in terms of known functions.
- Algorithmic methods. Under this heading, we include Risch integration, Rothstein-Trager-Rioboo integration, and others, which require extended computations.

A table of reduction rules can serve more roles than merely the database for an evaluation system; it can also serve as a repository for mathematical knowledge. Each rule can be annotated with information on its derivation, with references to the literature, and so on. An evaluation system can display transformations as they are used, for the information of users.

Here, we consider the repository of transformation rules for inde nite integrals that is described in [5, 6]. We shall refer to it by the acronym Rubi: RUle-Based Integrator. We review the general state of the repository and then focus on particular aspects, namely, its e ciency, and the selection of output forms. Procedures have been written in Mathematica to implement the evaluation of integrals using the repository, and these procedures have been the basis of testing and comparisons.

The construction and selection of the rules is based on the principle of mutual exclusivity. For a database of reduction rules to be properly defined, at most one of the rules can be applicable to any given expression. Mutual exclusivity is critical to ensuring that rules can be added, removed or modified without affecting the other rules. Such stand-alone, order-independent rules make it possible to build a rule-based repository of knowledge incrementally and as a collaborative effort.

3 Performance Comparison with Other Systems

In order to provide quantitative evidence of the benefits of rule-based integration, we present a comparison of the performance of various computer algebra systems on a test suite containing 7927 problems. The performance measure is was developed using the test suite, its good performance is to be expected, but even so, the favourable comparison with the other systems remains valid.

Although the test suite of 7927 problems is large, the problems themselves

Fig. 1. The node count for expressions returned by Mathematica 7 for the integral in (1). The horizontal axis shows values of the exponent m, while the vertical axis shows the node count for the corresponding expression for the integral.

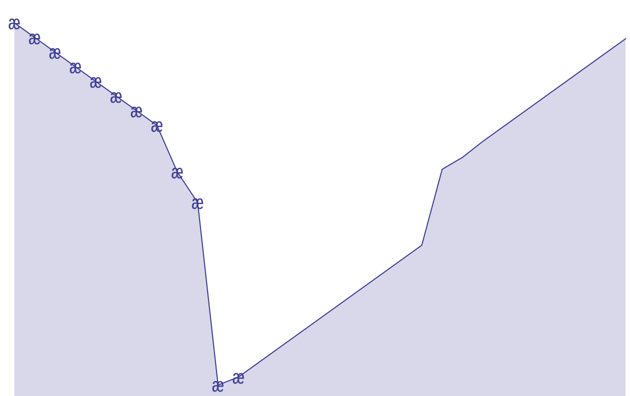

Fig. 2. The node count for expressions returned by Maple 13 for the integral in (1). The horizontal axis shows values of the exponent m, while the vertical axis shows the node count for the corresponding expression for the integral.

3. N: $bc - ad = 0,\ m + n + 1 = 0$

T: $\int (a + bx)^m (c + dx)^n \,\mathrm{d}x \to (a + bx)^{m+1} (c + dx)^n \ln(a + bx)/b$.

4. N: $bc - ad = 0,\ m + n + 1 \neq 0$

T: $\int (a + bx)^m (c + dx)^n \,\mathrm{d}x \to \dfrac{(a + bx)^{m+1} (c + dx)^n}{b(m + n + 1)}$.

5. N: $bc - ad \neq 0$

T: $\int (a + bx)^{-1} (c + dx)^{-1} \,\mathrm{d}x \to \dfrac{\ln(a + bx) - \ln(c + dx)}{bc - ad}$.

6. N: $bc - ad \neq 0,\ m + n + 2 = 0,\ n \neq -1$

T: $\int (a + bx)^m (c + dx)^n \,\mathrm{d}x \to -\dfrac{(a + bx)^{m+1} (c + dx)^{n+1}}{(n + 1)(bc - ad)}$.

7. N: $m + n + 1 = 0,\ m > 0,\ bc - ad \neq 0$

T: $\int (a + bx)^m (c + dx)^n \,\mathrm{d}x \to -\dfrac{(a + bx)^m}{dm(c + dx)^m}$

$$+ \frac{b}{d} \int (a + bx)^{m-1} (c + dx)^{-m} \,\mathrm{d}x .$$

8. N: $bc - ad \neq 0,\ m + n + 1 \neq 0,\ n > 0$

T: $\int (a + bx)^m (c + dx)^n \,\mathrm{d}x \to \dfrac{(a + bx)^{m+1} (c + dx)^n}{b(m + n + 1)}$

$$+ \frac{n(bc - ad)}{b(m + n + 1)} \int (a + bx)^m (c + dx)^{n-1} \,\mathrm{d}x$$

2. It should be noted that rule 6 is in fact a special case of rule 9. It is included because it is convenient to have an explicitly non-recursive entry.
3. Rules 8 and 9 respectively increment and decrement one of the exponents of the integrand. Unlike the other rules, it is not always obvious which of these two rules should be applied to a given integrand in order to minimize the number of steps required to integrate it. This choice is the subject of our optimization.

5 Integration strategies

The rules stated above describe a complete strategy for integration of the given class of integrals. The strategy is not unique, however, and other strategies might be more efficient. We therefore describe two other strategies and compare them with the preferred strategy.

5.1 Preliminary strategy 1

We replace rule 8 with a rule 8a, in which the simplification conditions are removed. Thus we have

8a. N: $bc - ad \neq 0$, $m + n + 1 \neq 0$, $n > 0$

T: $$\int (a+bx)^m (c+dx)^n \,\mathrm{d}x \to \frac{(a+bx)^{m+1}(c+dx)^n}{b(m+n+1)} + \frac{n(bc-ad)}{b(m+n+1)} \int (a+bx)^m (c+dx)^{n-1} \,\mathrm{d}x\,.$$

The effect of removing the restrictions is that all integrals will be reduced until one of the exponents becomes zero, at which point rules 1 to 6 will terminate the reduction. When this strategy is applied to the test case (1), the sizes of the results are as shown in figure 3.

The dip for the case $m = 10$ is important. For this case, rule 6 provides a direct one-step integration to a very compact form:

$$\int \frac{x^{10}\,\mathrm{d}x}{(1+x)^{12}} = \frac{x^{11}}{11(1+x)^{11}}\,.$$

This possibility is not noticed by the standard integrators of Mathematica and Maple, as can be seen in figures 1 and 2.

5.2 Preliminary strategy 2

We now remove the restrictions from rule 9, and place it above rule 8. Thus the rule becomes

9a N: $bc - ad \neq 0$, $n + 1 \neq 0$

T: $$\int (a+bx)^m (c+dx)^n \,\mathrm{d}x \to -\frac{(a+bx)^{m+1}(c+dx)^{n+1}}{(n+1)(bc-ad)} + \frac{(m+n+2)b}{(bc-ad)(n+1)} \int (a+bx)^m (c+dx)^{n+1} \,\mathrm{d}x$$

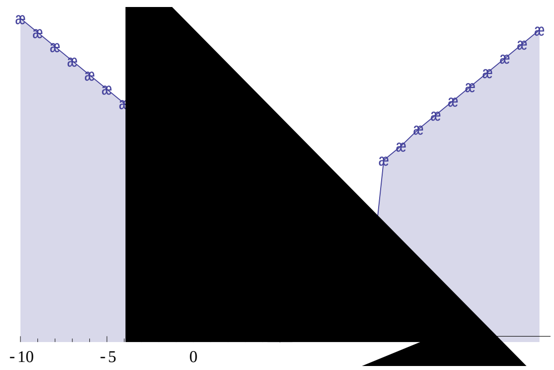

Fig. 3. The node count for expressions returned by the first alternative integration strategy for the integral in (1). The horizontal axis shows values of the exponent m, while the vertical axis shows the node count for the corresponding expression for the integral.

The effect of this is to increase one negative exponent until rule 6 can be applied. The resulting statistics on the size of integral expressions is shown in figure 4.

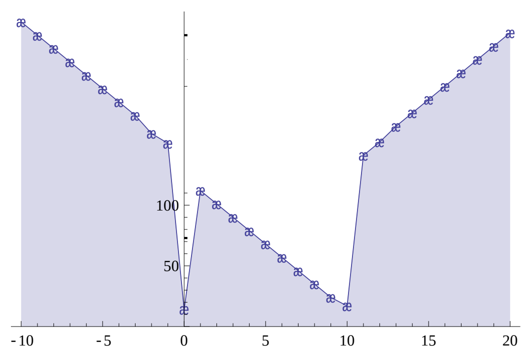

Fig. 4. The node count for expressions returned by the second alternative integration strategy for the integral in (1). The horizontal axis shows values of the exponent m, while the vertical axis shows the node count for the corresponding expression for the integral.

The dip at $m = 0$ is a result of rule 2 being applied before the general rules.

5.3 An optimal strategy

Clearly, one can obtain smaller expression sizes if one can switch between the two strategies just tested. This is what is done in rules 8 and 9 as presented. For

known solutions are $J(0; N; P)$ and $J(M; N; N)$. It is straightforward to derive the equality

$$J(m; n; p) = (1/b)J(m - 1; n + 1; p) - (a/b)J(m - 1; n; p) \quad (4)$$

An obvious strategy for $m > 0$ is to use this relation to reduce all integrals to the form $J(0; N; p)$. Thus, using the above conventions for describing a reduction rule, the rule reads

11. T: $J(m; n; p) \rightarrow \frac{1}{b}J(m - 1; n + 1; p) - (a/b)J(m - 1; n; p)$
    S: $m \in \mathbb{Z}$, $m > 0$, $n; p \in \mathbb{Q}$, $n - p < 0$

Next, Mathematica:

$$= \frac{1}{500}\left(\frac{20(-5+3x)\sqrt{2+3x}}{x\sqrt{5-x}} + 21\sqrt{10}\ln\left(21\sqrt{10}\,x\right)\right.$$
$$\left. - 21\sqrt{10}\ln\left(50\left(20+13x+2\sqrt{10}\sqrt{5-x}\sqrt{2+3x}\right)\right)\right)$$

Finally, Maple:

$$= \frac{1}{500}\left(21\sqrt{10}\,\mathrm{arctanh}\left(1/20\,\frac{(20+13x)\sqrt{10}}{\sqrt{10+13x-3x^2}}\right)x^2\right.$$
$$- 105\sqrt{10}\,\mathrm{arctanh}\left(1/20\,\frac{(20+13x)\sqrt{10}}{\sqrt{10+13x-3x^2}}\right)x + 60x\sqrt{10+13x-3x^2}$$
$$\left. - 100\sqrt{10+13x-3x^2}\right)\sqrt{5-x}\sqrt{2+3x}\,(-5+x)^{-1}\frac{1}{\sqrt{10+13x-3x^2}}\,x^{-1}$$

There is a disadvantage, however, to stepwise application of the above reduction, a disadvantage well known in other contexts. This is the repeated evaluation of the same integral during recursive calls. The standard example of this effect is the recursive evaluation of Fibonacci numbers. This is paralleled in applications of (4) and (6). This effect was one reason that Maple introduced its `option remember` early in its development. The important additional feature present here, that is not present in the Fibonacci example, is the possibility of different simplification options directing the computation to simpler results.

7 Concluding remarks

In [5], a number of advantages of rule-based simplification were listed. These included (see reference for details).

- Human and machine readable.
- Able to show simplification steps.
- Facilitates program development.
- Platform independent.
- White box transparency.
- Fosters community development.
- An active repository.

In this paper we have shown that an additional advantage of rule-based evaluation, illustrated in the integration context, is greater simplicity of results. Finally, we wish to point out that the integration repository described here has been published on the web [6], and is available for viewing and testing by all interested people.

References

1. Abramowitz, M. & Stegun, I., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. US Government Printing O ce, 1964. 10th Printing December 1972.
2. Fateman, Richard J., A Review of Mathematica, J. Symb. Computation, 13(5), 1992.
3. Einwohner, T.H. & Fateman, Richard J., *Searching techniques for integral tables*, Proceedings ISSAC '95, pp 133-139, ACM Press, 1995.
4. Gradshteyn, I.S. & Ryzhik, I.M., Table of Integrals, Series and Products. Academic Press 1965.
5. Rich, Albert D. & Je rey, David J., *A knowledge repository for inde nite integration based on transformation rules*. In: Intelligent Computer Mathematics, edited by J. Carette, L. Dixon, C.S. Coen, S.M. Watt. LNAI 5625, pp 480{485, Springer 2009.
6. Rich, Albert D. *Inde nite Integration Rules*, `www.apmaths.uwo.ca/RuleBasedMathematics`

Test Items		Rubi: Rule-based Integrator			
Integrand	Number	Optimal	Messy	Inconc.	Invalid
Rational	1426	1424	1	1	0
Algebraic	1494	1483	8	3	0
Exponential	456	452	0	4	0
Logarithmic	669	667	0	2	0
Trigonometric	1805	1794	8	3	0
Hyperbolic	1386	1379	6	1	0
Inverse trig	283	281	0	2	0
Inverse hyperbolic	342	335	2	5	0
Special functions	66	66	0	0	0
Percentages		99.4%	0.3%	0.3%	0%

Table 1. The integration test suite, with the numbers of problems broken down in categories. The performance of the Rule-based Integrator (Rubi) is given using measures described in the text.

Test Items		Maple			
Integrand	Number	Optimal	Messy	Inconc.	Invalid
Rational	1426	1176	249	0	1
Algebraic	1494	1126	277	45	46
Exponential	456	351	63	37	5
Logarithmic	669	284	161	194	30
Trigonometric	1805	1054	619	83	49
Hyperbolic	1386	521	641	181	43
Inverse trig	283	206	64	5	8
Inverse hyperbolic	342	159	96	55	32
Special functions	66	38	1	25	2
Percentages		62.0%	27.4%	7.9%	2.7%

Table 2. The performance of Maple on the test suite, using measures described in the text.

Test Items		Mathematica			
Integrand	Number	Optimal	Messy	Inconc.	Invalid
Rational	1426	1239	187	0	0
Algebraic	1494	1228	246	18	2
Exponential	456	406	32	12	6
Logarithmic	669	581	84	4	0
Trigonometric	1805	1212	573	3	17
Hyperbolic	1386	911	464	6	5
Inverse trig	283	211	62	10	0
Inverse hyperbolic	342	198	140	3	1
Special functions	66	53	9	4	0
Percentages		76.2%	22.7%	0.8%	0.4%

Table 3. The performance of Mathematica on the test suite, using measures described in the text.