Numerical Evaluation of Airy Functions with Complex Arguments

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Received June 30, 1989; revised June 4, 1990

We present two methods for the evaluation of Airy functions of complex argument. The first method is accurate to any desired precision but is slow and unsuitable for fixed-precision languages. The second method is accurate to double precision (12 digits) and is suitable for programming in a fixed-precision language such as FORTRAN. The first method uses the symbolic manipulation language Maple to evaluate either the Taylor series expansion or an asymptotic expansion of each function. The second method extends an idea of J. C. P. Miller to the complex plane. It uses the first method to obtain a grid of points

The Airy functions appear in the solution of several problems in fluid mechanics, geophysics, and atomic physics. We now briefly discuss some of these. More details can be found in the cited references.

For plane Couette flow it can be shown (Drazin and Reid [4]) that the Orr–Sommerfeld equation for linear stability has the form

$$(\varepsilon^3(D^2-\alpha^2)-\eta)(D^2-\alpha^2)\phi=0 \qquad (1.2)$$

[redacted]

boundary conditions on $c(k, z)$ are

$$Dc = -1 \quad \text{at} \quad z = 0, \; c \to 0 \text{ as } z \to \infty.$$

tions of (2.1), they are necessarily linearly dependent on $Ai(z)$ and $Bi(z)$. Explicitly (Abramowitz and Stegun [12], we have

[redacted]

of very high precision, limited only by the memory of the machine used. Either of these two facilities enables us to use a very simple brute force approach to the evaluation of the to find a bound on the truncation error.

To bound the truncation error we start with the recurrence relation (3.1) and the definition of the Taylor series sum for either $S = f(x)$ or $S = g(z)$:

$$S = \sum_{k=0}^{\infty} Y_k = \sum_{k=0}^{n} Y_k + \sum_{k=n+1}^{\infty} Y_k. \tag{3.4}$$

If we take n so $Y_n \neq 0$, then by the construction of the series for f or g, $Y_{n+1} = Y_{n+2} = 0$. Therefore,

$$S = \left(\sum_{k=0}^{n} Y_k\right) + Y_{n+3} + Y_{n+6} + Y_{n+9} + \cdots$$

$$\frac{|\Delta Bi(z)|}{|Bi(z)|} \leqslant \frac{(Bi(0) + |z|\, Bi'(0))\, |F_{n+1}|}{\left((1 - r_{n+1})\, |\hat{B}i(z)| - (Bi(0) + |z|\, Bi'(0))\, |F_{n+1}|\right)}. \tag{3.6}$$

A Maple routine has been written to calculate $Ai(z)$, $Ai'(z)$, $Bi(z)$, and $Bi'(z)$ by Taylor series and to return error estimates based on the above formulae. Note that the idea of numerical linear dependence plays a smaller role in the very high precision context: one need only take enough figures and the mathematical linear independence is evident.

3.2. *Summation of Asymptotic Series for Large* $|z|$

It is clear that the cost of computing $Ai(z)$ to a given as

and $Bi'(z)$, we may investigate more efficient, fixed-precision algorithms. Here we develop a method, also based on usually impossible. The problem is unavoidable in this case.

4.1 *Evaluation Using a Triangular Grid*

and an absolute tolerance must be used here instead.

[redacted]

Likewise, to calculate $Bi(z)$ we use the Taylor expansion about the upper left point, again so that we calculate in a direction giving exponential growth of the desired component.

4.2. *Truncation Error Analysis for the Modified Taylor*

One advantage of the Taylor series is that we can obtain a useful (for diagnostic purposes) bound on the truncation error. We saw earlier a bound on the cancellation error for the Taylor series. After the truncation error bound for the Taylor series is given, we shall derive similar results for the cancellation error in the asymptotic series.

We compute $y(z) = y(c + z - c) = y(c + h)$ via

$$(\rho')^3 = 1 + \varepsilon\,|c|\,\rho'$$

and this cubic is obviously numerically stable to solve for small ε. Then

$$a_k \leqslant (|\alpha\rho_1^{n-2}| + |\beta\rho_2^{n-2}| + |\gamma\rho_3^{n-2}|)\,\rho^{k-(n-2)},$$

where $\rho = \max(|\rho_1|, |\rho_2|, |\rho_3|)$, and instead of finding α, β, γ, we solve the more stable linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 \\ \rho_2^1 & \rho_2^2 & \rho_3^3 \end{bmatrix} \begin{bmatrix} \alpha\rho_1^{n-2} \\ \beta\rho_2^{n-2} \\ \gamma\rho_3^{n-2} \end{bmatrix} = \begin{bmatrix} |A_{n-2}| \\ |A_{n-1}| \\ |A_n| \end{bmatrix}$$

(which is a system of Vandermonde type). Finally,

[redacted]

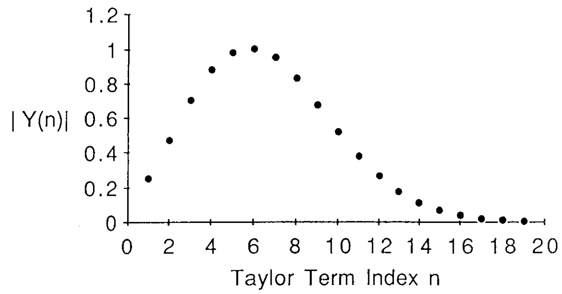

FIG. 3. The magnitude of the complex Taylor series terms for a Taylor stepsize of magnitude 4, which, for emphasis, is larger than any actually used by the program.

our Taylor series coefficients had in fact been minimal, the

results of these experiments are presented in Fig. 3 and 4. The propagation of errors in the recurrence relation for y_n depends in part on the magnitudes $|y_n|$. If these terms are large, initial errors are amplified by these factors. We see in Fig. 3 the magnitudes of a typical sequence of Taylor series terms, with magnitudes at most $O(1)$, which cause no difficulty. Of course, for large h the height of the "hump" increases, and eventually errors could become serious. For the values of h used by our program (everywhere $|h| \leqslant$ maximum triangle diameter ~ 2.5). This is not a problem. In the graph presented, $|h| = 4$ which is larger than that used in the program, for emphasis. In Fig. 4, we observe the error amplification factors due to a perturbation in c, the base point about which we are Taylor expanding, for the

TABLE II

Zeros of $Bi(z)$ in Upper Half Plane

0.97754488673162E + 00	0.21412907060387E + 01

ACKNOWLEDGMENTS

This work was supported financially in part by the Natural Sciences and

Maple group at the University of Waterloo provided a β-test version of