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Polynomial Transformations of Tschirnhaus, Bring and Jerrard

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Abstract

Tschirnhaus gave transformations for the elimination of some of the intermediate terms in a polynomial. His transformations were developed further by Bring and Jerrard, and here we describe all these transformations in modern notation. We also discuss their possible utility for polynomial solving, particularly with respect to the Mathematica poster on the solution of the quintic.

1 Introduction

A recent issue of the BULLETIN contained a translation of the 1683 paper by Tschirnhaus [10], in which he proposed a method for solving a polynomial equation $P_n(x)$ of degree n by transforming it into a polynomial $Q_n(y)$ which has a simpler form (meaning that it has fewer terms). Specifically, he extended the idea (which he attributed to Decartes) in which a polynomial of degree n is *reduced* or *depressed* (lovely word!) by removing its term in degree $n-1$. Tschirnhaus's transformation is a polynomial substitution $y = T_k(x)$, in which the degree of the transformation $k < n$ can be selected. Tschirnhaus demonstrated the utility of his transformation by apparently solving the cubic equation in a way different from Cardano.

Although Tschirnhaus's work is described in modern books [9], later works by Bring [2, 3] and Jerrard [7, 6] have been largely forgotten. Here, we present the transformations in modern notation and make some comments on their utility. The investigations described here were stimulated by work on the Quintic poster [11] and therefore the discussion is directed towards the quintic. We shall deal with the following forms of a quintic:

$$x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0\,; \qquad \text{General quintic form} \tag{1}$$
$$x^5 + b_3 x^3 + b_2 x^2 + b_1 x + b_0 = 0\,; \qquad \text{Reduced quintic form} \tag{2}$$
$$x^5 + c_2 x^2 + c_1 x + c_0 = 0\,; \qquad \text{Principal quintic form} \tag{3}$$
$$x^5 + d_1 x + d_0 = 0\,: \qquad \text{Bring—Jerrard quintic form} \tag{4}$$

2 Tschirnhaus's solution of the cubic

Before considering the quintic, we look at the cubic. Tschirnhaus transformed the depressed cubic $x^3 + px + q = 0$ into the binomial $y^3 + r = 0$. This can be done efficiently using the resultant, which is a tool from after Tschirnhaus's time, of course. Let $P_3(x)$ be the cubic and let $T_2(x; y)$ be Tschirnhaus's transform, then we have

$$P_3(x) \quad = \quad x^3 + px + q = 0\,; \tag{5}$$

$$T_2(x; y) \quad = \quad x^2 + \text{®}x + \tfrac{2}{3}p + y = 0\,; \tag{6}$$

$\mathrm{res}_x(P_3$

4 Transformation to a Bring-Jerrard form

Removing, from a general quintic, the three terms in x^4, x^3 and x^2, brings it to Bring—Jerrard form. Tschirnhaus clearly thought that he would be able to do this by using the cubic transformation

$$z_k = {x_k}^3 + ® {x_k}^2 + ¯ x_k + ° : \tag{15}$$

This section shows that it is not always possible to solve for the coefficients $®; ¯; °$ in terms of radicals, which are the quantities that Tschirnhaus would have been expecting to use.

To reduce the size of the equations, we shall start with a principal form cubic $x^5 + c_2 x^2 + c_1 x + c_0 = 0$ and eliminate the x^2 term. Extending the approach of the previous section in the obvious way, we use the power sums for the new quintic

$$\begin{aligned} S_1(z_k) &= S_2(z_k) = S_2(z_k) = 0; \\ S_4(z_k) &= -4d_1; \\ S_5(z_k) &= -5d_0: \end{aligned} \tag{16}$$

From the first three equations in (16) we determine $®; ¯; °$. The remaining two equations will determine the new coefficients. Using (15) to evaluate $S_1; S_2; S_3$, we obtain three equations for the parameters:

$$5° - 3c_2 = 0; \tag{17}$$

$$5°^2 - 10®c_0 - 4®^2 c_1 - 8¯ c_1 - 6®¯ c_2 - 6° c_2 + 3c_2^2 = 0; \tag{18}$$

$$5°^3 - 15®^2¯ c_0 - 15¯^2 c_0 - 30®° c_0 - 12®¯^2 c_1 - 12®^2° c_1 - 24¯° c_1 + 9c_0 c_1 + 12®c_1^2 - 3¯^3 c_2 - 18®¯° c$$

Finally, setting the sum of the cubes of (20) to zero, $S_3(z_k) = 0$, we obtain a cubic equation for °. The equation will not be shown. Therefore all of the intermediate quantities can be found in terms of radicals.

This work was generalized by Jerrard [7, 6] (it is most likely that Jerrard was not aware of Bring's research) to show that the transformation (20) could be applied to a polynomial of degree n to remove the terms in x^{n-1}, x^{n-2} and x^{n-3}. In particular, Jerrard claimed that he developed a method for solving a general quintic. Hamilton was asked to report on this. In his detailed report, Hamilton showed that Jerrard had not completely solved the general quintic equation by radicals. The report is available at
`http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Jerrard/`.

5 Discussion

Starting with a general quintic (1), the transformation (20) produces the Bring–Jerrard form. In the case of a solvable