+ 3)

(CAS) are having an influence on many traditional formulae. This article sets down some of the ways in which computers shape our formulae, rough hew them how we will.

Let us see three examples immediately.

- Some CAS give the following formula:

$$\int x^n\,dx = \frac{x^{n+1}-1}{n+1} \ . \tag{1}$$

What is that “-1” doing in the numerator?

- Abramowitz & Stegun [AS65] give the solution of $x^3+3px-2q=0$ as

$$x = (q+(q^2+p^3)^{1/2})^{1/3} + (q-(q^2+p^3)^{1/2})^{1/3} \ , \tag{2}$$

but Maple and Mathematica both give the much uglier formula

$$x = {}^{3}\ q + \mathrm{p} \qquad (2)$$

1.1 Solving Problems using Formulae

No discussion of formulae can continue for very long without an irascible numerical analyst interjecting “You people should *not be using* a formula in the first place.” Having acknowledged the limitations of formulae, this article focusses on making formulae as useful as they can be. Alternative routes to problem solving are left to others. One final remark that must be made, however, is that formulae have more uses than just being a basis for numerical computation. They can also be used as the basis for a proof or a basis for insight.

2 Contrapuntus

Here are the themes of the paper. Each topic is discussed using more examples than there is room for in this extended abstract.

2.1 Special Cases

A formula can be expressed in algebraically equivalent ways, which, however, have different behaviour on substituting special values. Consider the two trigonometric identities

$$\arcsin z = 2\arctan\left(\frac{1-\sqrt{1-z^2}}{z}\right) \tag{5}$$

$$\arcsin z = 2\arctan\left(\frac{z}{1+\sqrt{1-z^2}}\right) . \tag{6}$$

Most people would prefer the first identity over the second, but substituting $z = 0$ into both shows that the first contains a removable singularity. Any user of formula (5) can replace the evaluation at $z = 0$ with the limit calculation $\lim_{z\to 0}(1-\sqrt{1-z^2})/z = 0$, but clearly the second identity is more efficient.

2.2 Definite Notation

It is a mistake to suppose that everyone agrees on the meaning of $\sqrt{x}$, or any other symbol. This problem has been discussed in [BCD+02]

2.3 Domains of Correctness

Companies selling computer algebra software constantly receive complaints from users that their system says

$$\int 1/x = \ln x .$$

Where are the absolute value signs? The user and the system are aiming at different domains of correctness.

2.4 Continuity

In addition to a formula having inconvenient points, it can fail to have an appropriate limit. Thus consider

$$\int x^{\varepsilon-1}\,\mathrm{d}x = x^{\varepsilon}/\varepsilon\ . \tag{7}$$

Taking the limit $\varepsilon \to 0$ is possible with the left-hand side, but not the right.

2.5 Numerical Accuracy

Every textbook on numerical analysis takes a shot at the quadratic formulae, and warns against numerical errors that they can cause [PTVF92] [Rec00]. The most common activity of users of formulae is to instantiate them, which is to say the user substitutes numerical values for the coefficients and then evaluates the formula. It is a standard topic in numerical analysis texts to discuss the rounding errors introduced during the evaluation of formulae. For this paper, the question is whether to use formulae that are numerically robust, or whether to use formulae that are attractive symbolically.

3 Conclusions

This brief outline needs more examples and discussion to be convincing, but it shows the main point, which is that those working in computer algebra development have made discoveries that have not been appreciated by the general mathematical community. A referee of this submission expressed the common assumption that there must be an established literature on these questions. I contend that these questions have not been written about as much as is needed. To rectify this situation we must do 2 things: identify our discoveries and write about them for a general mathematical audience. One example is the recent publication [JN04]. I hope my talk will stimulate some ideas about this.

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David Jeffrey. I obtained my Ph.D. in fluid mechanics from Cambridge University in 1974 and discovered computer algebra while working on perturbation solutions of the Stokes equations. The first system I used was CAMAL. On moving to Canada I started using MAPLE and DERIVE. The proximity of Waterloo (MAPLE's home) led me to vent my frustrations with early versions of MAPLE at their research meetings, and as a result my research drifted into finding remedies for some of MAPLE's shortcomings. Around the same time, I also started working with Albert Rich, the author of DERIVE. The arrive of Stephen Watt at UWO and the establishing of the ORCCA lab increased my involvement in computer algebra research. However, I still think of myself has having "a foot in each camp", and continue to publish occasionally on topics in mechanics.

djeffrey@uwo.ca www.apmaths.uwo.ca/˜djeffrey