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## Chapter 4

# Some basic principles in interaction calculations

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### Abstract

Some general aspects of the interactions taking place within a suspension can be understood by exploiting properties of the Stokes equations; these properties and their applications are described. Specifically, the way in which reversibility can be used to predict overall properties of a flow is explained by analysing several applications both informally and formally; Faxén's laws for the response of a particle to an ambient flow are examined to clarify common conceptual difficulties; and the use of lubrication theory to approximate interactions between close particles is developed carefully. In addition, the ways in which tensors can be used to summarize interaction results are covered, and the principles are illustrated whereby tensor relations can be simplified by appealing to geometrical symmetries and the character of tensor transformations.

### Nomenclature

$a$	Radius of spherical or circular particle
$\mathbf{A}, \mathbf{B}$	Resistance tensors
$\mathbf{E}$	Rate-of-strain tensor
$\mathbf{f}$	Equivalent surface forces
$\mathbf{F}, F_i$	Force on particle
$\mathbf{g}$	Acceleration due to gravity
$\mathbf{G}$	Green's function for Stokes flow
$G_{ijk}$	Resistance tensor

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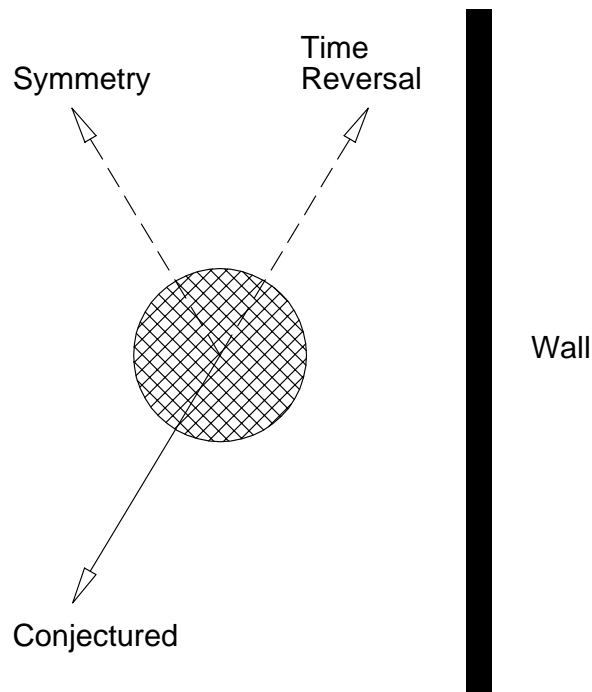
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about the interactions between suspended particles at low Reynolds numbers than it is at higher Reynolds numbers

obtained its position and velocity; the same flow will be found whether the sphere is moving at constant velocity  $\mathbf{V}$  or whether it is undergoing an oscillatory motion and has the velocity  $\mathbf{V}$  only at that instant. Consequently, if time is reversed, there is no change in the governing equations, and the system retraces its steps, moving in a valid Stokes flow.

Some visually compelling demonstrations of reversibility can be seen in the film *Low-Reynolds-number flows* that was made by G.I. Taylor in 1967\*. One demonstration, for instance, uses very viscous oil in the annulus between two concentric vertical cylinders, the inner cylinder being able to rotate about its vertical axis. Some dye is injected into the oil and then the cylinder is rotated, causing the dye to smear out, or so it seems. When the motion is reversed, however, all the dye returns to its starting position (except for a little blurring due to molecular diffusion). A similar demonstration, using the same apparatus, places a rigid body in the oil, and again the inner cylinder is rotated. The body translates and rotates away from its initial configuration, but when the whole motion is reversed, the body returns to its starting point. Another demonstration shows a small mechanical fish vainly trying to swim by flapping a tail. At high Reynolds number, it moves forward, but at low Reynolds number, each time the tail reverses its motion, so does the fish.

The above demonstrations require reversibility alone; the next one combines reversibility with another symmetry possessed by the flow. Consider a sphere falling parallel to a vertical plane wall. The fact that the sphere falls at a constant distance from the wall is demonstrated in the following way. First one imagines that the sphere falls for a short time downwards. For the sake of setting up a contradiction, it is necessary to conjecture that it is *not* the case that the sphere stays at a constant distance from the wall, and that instead it moves away from the wall. Now consider what would happen if time were to run backwards. Clearly the sphere would retrace its path and move closer to the wall as it rose. Now observe that a second way to get the sphere to move upwards is to reverse gravity while leaving time running forward. Because the wall is a plane, the flow situation after gravity has been reversed is identical to the situation before (notice that this geometrical symmetry is independent of the reversibility just described). Accordingly, if the sphere is subjected to an upward force, then, by the starting conjecture, it will move away



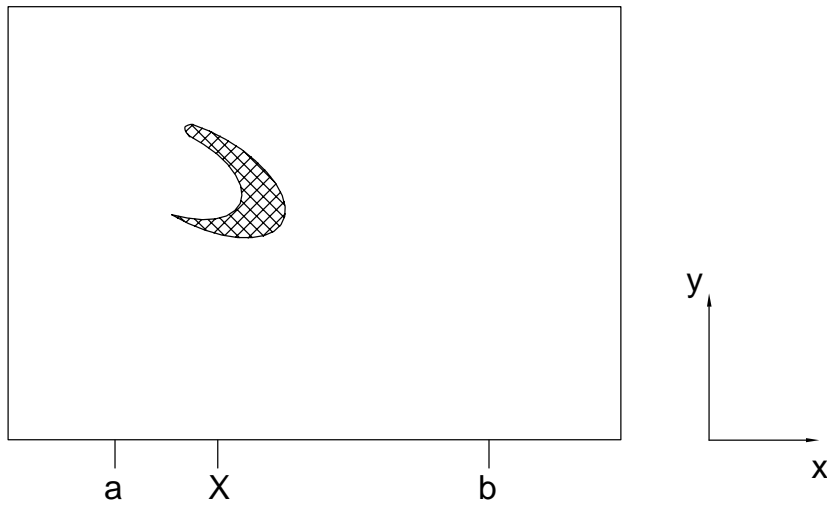
**Figure 1.** A sphere falling, or rising, next to a plane wall.

from the wall as it rises, as shown in figure 1.

This is a contradiction: on the one hand, geometrical symmetry says that an upward-moving sphere will move away from the wall, while reversibility says that the sphere will move towards the wall. The conclusion is that it will neither move closer nor away, but stay at the same distance.

Reversibility is also important for periodic motion. Numerous periodic motions have been found in systems of particles moving in Stokes flow, and their existence can have a strong influence on calculations. If the system is scleronomic, meaning the external forces on the flow do not change with time, then a moving system of particles that passes through a configuration twice must be executing a periodic motion, and reversibility combined with another symmetry can prove this, as the tumbling of an ellipsoidal particle in a shear flow illustrates. The ellipsoid executes a closed periodic orbit, and this orbit is described by an orbital constant, whose value depends upon how the particle starts its motion. One might suppose that the ellipsoid could follow an orbit that is not exactly peri-





**Figure 2.** A simplified version of Taylor's demonstration.

with respect to its centroid at  $\mathbf{x}_c$ . The boundary condition on its surface is then

$$\mathbf{u} = \mathbf{V} + \boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{x}_c), \quad (2)$$

and in addition  $\mathbf{V}$  and  $\boldsymbol{\Omega}$  are chosen so that the total force  $\int \boldsymbol{\sigma} \cdot \mathbf{n} \, dS$  on the particle is zero, as is the total moment  $\int \boldsymbol{\sigma} \cdot \mathbf{n} \times (\mathbf{x} - \mathbf{x}_c) \, dS$ . Here  $\boldsymbol{\sigma}$  is the stress tensor. It will now be proved that, for the purpose of calculating the displacements of fluid particles, it does not matter how the point  $X$  gets from  $x = a$  to  $x = b$ ; in other words, for all possible functions  $X(t)$ , the fluid elements and the suspended particle **Cauchy's Theorem** and the su  $V$   $T$   $n$   $\mathbf{d}$

condition is also satisfied, because, using subscript notation,

$$\begin{aligned}\sigma_{ij} &= -p\delta_{ij} + \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \\ &= V(t) \left( -p^* \delta_{ij} + \mu \left[ \frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right] \right) = V(t) \sigma_{ij}^* .\end{aligned}$$

Since  $\sigma_{ij}^*$  integrates to zero force, so does  $\sigma_{ij}$ . The other boundary conditions are satisfied in a similar manner.

The displacement of a fluid element is calculated by considering what happens during a time  $\delta t$ . If a fluid element is at  $\mathbf{x}_L(t)$  (the subscript is a reminder that  $\mathbf{x}_L$  is a Lagrangian quantity), then its motion is given by

$$\frac{d\mathbf{x}_L}{dt} = \mathbf{u}(\mathbf{x}_L(t), t) , \quad (3)$$

here  $\mathbf{u}$  is the Eulerian velocity that obeys the Stokes equations. Therefore, in time  $\delta t$ , the displacement of the fluid element is given approximately by  $\delta t \mathbf{u}(\mathbf{x}_L, t) = \delta t V(t) \mathbf{u}_L^*$ , using the results established above. Note  $\delta t V(t) = \delta x$ , the displacement of the boundary. Thus all fluid displacements during the motion are proportional to the corresponding displacement of the boundary, implying that the total displacement of any particular element is proportional to  $\int_0^T V(t) dt = X(T) - X(0) = b - a$ , whatever the function  $X(t)$ . This can be seen in the demonstration



part of the solution), together with  $\mathbf{t} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = 0$  here  $\mathbf{t}$  is a unit tangent vector to the surface. The equations and boundary conditions are again linear and scale with the boundary velocity.

The strong form of reversibility applies to systems in which the Stokes equations contain a single time scale, imposed by the boundaries through the boundary conditions. Such cases will appear in the next chapter.

containing very small particles will be subject to physical processes that modify the governing equations and destroy reversibility, such as electric charge effects, Brownian motion, and so on.

### 3 Faxén laws

Faxén laws are exact mathematical statements about the response of a particle to an ambient flow. They are used, however, mostly in the approximate calculation of interactions between particles, and this ambivalence can be a cause of confusion. Suppose that a region of fluid contains an ambient flow field  $\mathbf{u}^\infty(\mathbf{x})$ , and suppose that a rigid spherical particle is introduced into this field at a point  $\mathbf{x} = \mathbf{x}_0$ . The force  $\mathbf{F}$  and couple  $\mathbf{L}$  experienced by the sphere, which is stationary, are given exactly by

$$\mathbf{F} = 6\pi a\mu \left[ \mathbf{u}^\infty(\mathbf{x}_0) + \frac{1}{6}a^2 \nabla^2 \mathbf{u}^\infty(\mathbf{x}_0) \right], \quad (5)$$

and

$$\mathbf{L} = 8\pi a^2 \mu \nabla \times \mathbf{u}^\infty(\mathbf{x}_0). \quad (6)$$

Laws for other quantities are also known. It is certainly striking, on first acquaintance, to see that only the local velocity and its first and second derivatives are important, no matter how complicated the ambient flow may be. As Kim & Karrila (1991) point out, the interpretation can be simplified further by noting that  $\nabla^2 \mathbf{u}^\infty$  is proportional to the ambient pressure gradient. It is natural to contrast this simplicity with the lengthy calculations in the literature of the motion of a particle near a wall or a second particle, and to wonder how it can be exact. Students sometimes draw a diagram of a sphere in a maelstrom of streamlines, and ask whether Faxén laws apply to this situation.

An understanding of whether Faxén laws are exact or approximate can be obtained by reviewing a proof of them, here the proof that is given in the appendix to Batchelor (1972). Any solution of the ‘ $\nabla^2 \mathbf{u} = \mathbf{0}$ ’

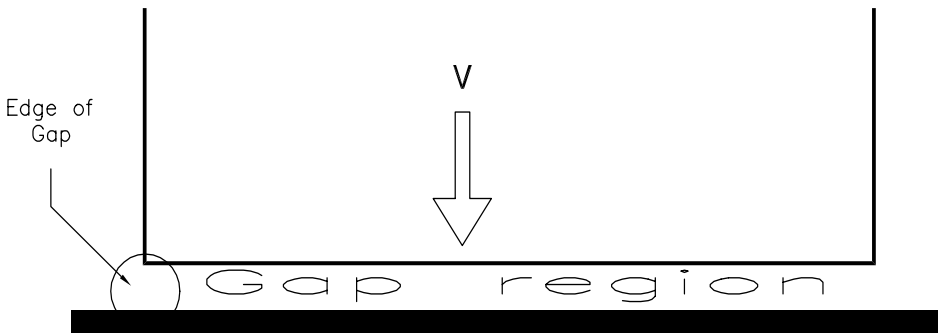
Then  $\mathbf{G}(\mathbf{x} - \boldsymbol{\xi}) \cdot (\mathbf{n} \cdot \boldsymbol{\sigma} \, dS)$  is the velocity field produced at  $\mathbf{x}$  by that force. The second term  $\mathbf{n} \cdot \boldsymbol{\nabla} \cdot \mathbf{u} \, dS$  corresponds to a 'double layer', and has a similar interpretation. The solution (7) can be interpreted by saying that the flow outside a rigid particle cannot tell the difference between a physical boundary and an appropriate distribution of point forces and double layers. Faxén's theorem is derived by adding such a distribution of forces to a region of fluid so

Some of the usual questions about Faxén laws can now be answered. The first question asks ‘Is it really exact?’. The answer is ‘Yes’, provided the limit  $R \rightarrow \infty$  is accepted. This limit is often taken in fluid mechanics, and although it can cause difficulties, for example uniform flow around a cylinder in two dimensions, it is usually accepted if  $\mathbf{u}^\infty$  is simple. The second question asks ‘Does it apply to flows with large curvature?’. Large curvature means that the values of  $\nabla\mathbf{u}^\infty$  and higher derivatives are large, in order for the streamlines of  $\mathbf{u}^\infty$  to be highly curved. Here the answer is ‘Technically yes, but in practice no’. It is technically yes, because it is possible mathematically to imagine a flow with large curvature produced by very distant boundaries. It is no in practice, because the person asking the question is almost certainly thinking of a situation in which the curvature is caused by boundaries fairly close to the particle. For example, the particle is in a curving pipe or near other particles. Thus the flow is probably not caused by large forces far away, but by ordinary forces that are close. In this case the questioner is really refusing to ignore the reflections from the boundary, in other words, refusing to accept the premise of the theorem.

## 4 Lubrication theory and close particles

The interactions between two nearly touching particles can often be calculated using *lubrication theory*. This ‘theory’ is actually a set of approximations that allow the Navier-Stokes equations to be simplified to a form in which they are more easily solved. In order for the approximations to be valid, the particles must be nearly touching, and in addition their relative motion must cause the fluid in the gap between them to be highly sheared. For example, if two close particles are in the process of moving past each other, then lubrication theory can be applied, but if they are sedimenting side-by-side with no relative motion between them, it cannot. Although the name lubrication theory is a reminder that the approximations were first worked out in the analysis of flows in lubricated bearings, the theory in the low-Reynolds-number context is not identical to its engineering namesake; some of the ways in which the approximations are developed in Stokes flow give the application a distinctive slant.

The aspects of the theory that will be explored are the definition of the gap between the particles, the handling of the edge of the gap, and the role of the fluid far from the gap. It is important to discuss these points



**Figure 3.** A plate approaching a wall.

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because the justification for the theory is usually informal; it is developed initially for a situation in which the approximations have a strong intuitive appeal, or for a situation in which they can be proved correct. Subsequently, the approximations are applied to situations in which the justification is less straightforward, and there is a danger that future calculations will continue to use them when they have really ceased to apply as *in situ*  $\forall n \exists i \in \mathcal{P}$

non-dimensionalized velocity components be given by  $\mathbf{u} = V(u, v)$ . If the pressure is given by  $\mu V p$ , the equations are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \quad (10)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial p}{\partial y}, \quad (11)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (12)$$

The no-slip boundary conditions require that

$$u(x, 0) = u(x, h) = 0 \quad \text{and} \quad v(x, 0) = 0 \quad \text{and} \quad v(x, h) = -1.$$

Since, from the boundary conditions,  $v$  goes from  $-1$  to  $0$  in a distance  $h$ , the term  $\partial v/\partial y$  will be approximately  $-1/h$ , and then, from the continuity equation (12), this implies that  $\partial u/\partial x \approx 1/h$ . This in turn implies  $u \approx x/h$ , which is consistent with our global estimate above. Since  $u$  grows with  $x$  while  $v$  does not, it must be that  $u \gg v$  for  $x \gg h$ , i.e. away from the centre of the gap. So the terms in (11) are much less than those in (10), and (11) can be neglected. Further,  $\partial p/\partial y$  can be set to zero in comparison with  $\partial p/\partial x$ , meaning that  $p$  is approximately independent of  $y$ . The assumption that pressure is approximately constant across a thin layer of fluid is also used in boundary-layer theory. In equation (10), the derivative  $\partial^2 u/\partial y^2$  must be of the order of  $Vx/h^3$ , since  $u$  changes from  $0$  at the walls to  $Vx/h$  in the flow. The estimate  $u \approx Vx/h$  also implies  $\partial^2 u/\partial x^2 \approx 0$ , so clearly  $\partial^2 u/\partial y^2 \gg \partial^2 u/\partial x^2$ , and therefore (10) can be approximated by

$$\frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx}, \quad (13)$$

the total derivative of  $p$  showing that it depends only on  $x$ . Inte

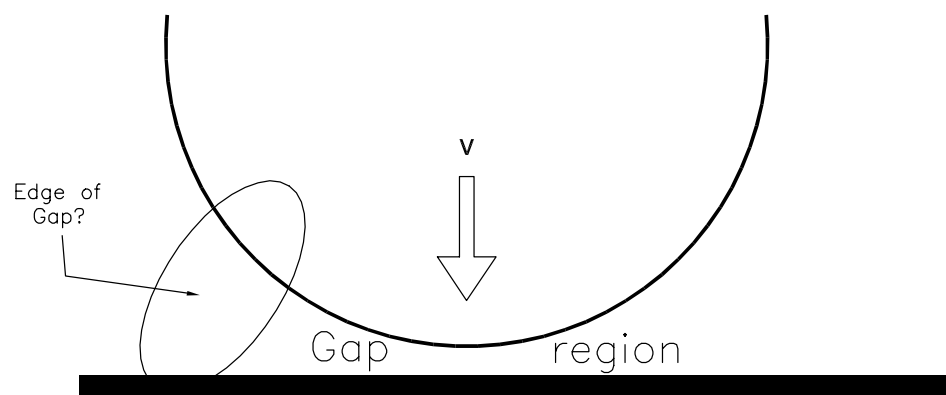
Therefore  $p = -6x^2/h^3 + Ax + B$ . The calculation is completed by applying boundary conditions on  $p$ . By the symmetry of the problem,  $A = 0$ , but what about  $B$ , the pressure at the centre of the gap?

As the plate approaches the plane, the fluid from the gap region will escape into the surrounding fluid, spreading out until it reaches the ambient pressure. Depending upon the Reynolds number of the flow outside the gap, this could even be a jet-like flow. The problem of how the fluid slows down on leaving the gap is a difficult calculation and it is better avoided if possible. Consequently, it is assumed that as soon as the fluid reaches  $x = \pm L$ , the pressure equals the pressure outside the gap (which is zero). This gives  $p = 6(L^2 - x^2)/h^3$ , and a non-dimensionalized force of  $2 \int_0^L p \, dx = 8L^3/h^3$ . The last assumption is in fact the specification of the edge of the gap (which was left vague above) and the reader's acceptance of it depends to some extent on mathematical outlook. The clear-cut geometry suggests strongly that the pressure will reach the ambient one within a distance  $O(h)$  from the edge, that is, within the circle labelled 'edge of gap' on figure 3. This will induce an error  $O(h/L)$ , which is of the same order as the approximations made in obtaining (13). It is possible to contrive flow conditions outside the gap that would invalidate the assumptions about the edge of the gap, but it is not a serious worry and the approximations above are well established.

The above example is important in engineering lubrication theory, but for sedimentation studies and other particulate interaction problems, the geometry is only generally relevant. Keeping with the simplification of two-dimensions, consider a cylinder approaching a plane, again with velocity  $V$ , as shown in figure 4. From looking at the figure, one can be convinced that there is a gap region between the cylinder and plane, although it is no longer possible to point to the edge of the gap with the confidence felt in the first example. One proceeds to analyse the flow in the gap, in the hope that everything will turn out all right in the end.

Let the radius of the cylinder be  $a$  and let the gap be  $h$  at its minimum. The velocities and pressure are again  $V(u, v)$  and  $\mu Vp$  and the approximations leading to (12) and (13) still apply. The boundary conditions can be simplified by expanding the expression for the cylinder surface for small values of  $x$ . Thus

$$y = a + h - \sqrt{a^2 - x^2} = h + \frac{x^2}{2a} + O(x^4) = H + O(x^4), \quad (15)$$



**Figure 4. A**



in some presentations, this limit is taken early in the treatment, so that the edge of the gap never appears.

The lubrication result (19) can be compared with the exact result obtained by Jeffrey & Onishi (1981). When their solution is expanded for small  $h$ , it becomes

$$F = 3\sqrt{2}\pi \left(\frac{h}{a}\right)^{-\frac{3}{2}} + \frac{63\sqrt{2}}{20} \left(\frac{h}{a}\right)^{-\frac{1}{2}} + O\left(\left(\frac{h}{a}\right)^{\frac{1}{2}}\right). \quad (20)$$

Both (19) and (20) contain the same leading term, but (20) contains a second singular term. Numerically, both terms are important: if  $h = 0.1$ , then the force according to the exact solution is 148.31, whereas one term of (15) gives 134.2 and two terms give 148.25. So the comparison provides reassurance on one point, but raises another. In particle geometries, the edge of the gap can be avoided to leading order, but lubrication theory must be extended to higher order to capture all of the important terms. Can the edge of the gap be ignored at higher order? This question has been investigated in three-dimensional flows.

The change from two dimensions to three dimensions is in the sense that the force is



The edge of the gap appears in the expression for the force in a way that cannot be removed by taking the limit  $R_G \rightarrow \infty$ , and this is the feature that this exampl

One of the attractions of lubrication theory is the fact that it can yield important information about interactions from a relatively simple analysis of only one part of the flow. If the edge of the gap cannot be removed from the calculations, lubrication theory loses some of its appeal. Thus the importance of the O'Neill & Stewartson result lies in the fact that it provides a foundation for other lubrication calculations to draw on. Even if the edge of the gap remains in the result of a gap calculation, it can be removed by postulating that a full analysis would find a cancelling term in the solution outside.

In a number of publications, lubrication theory has been pushed even further. Thus the approximations above have been carried

found good agreement with experiment. The motion of one sphere approaching a much larger one has been followed very accurately by Lecoq

The need for axial vectors is demonstrated by considering a rotating rigid body in a very simple case: a rigid body rotating around the  $z$  axis with vector angular velocity  $\boldsymbol{\omega} = \Omega \mathbf{k}$ . Consider velocities in the  $XY$  plane. The velocity  $(v_x, v_y)$  at any point  $(x, y)$  is given by  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ , meaning its components are given by

$$v_x = -\Omega y, \quad v_y = \Omega x. \quad (31, 32)$$

First consider the transformation to a dashed coordinate system defined by  $x' = y$ ,  $y' = -x$  and  $z' = z$ , which is a rotation about the  $z$  axis. The matrix  $a_{ij}^{(1)}$  for this transformation is

$$a^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

The matrix  $a^{(1)}$  can be used to transform both  $\boldsymbol{\omega}$ , which remains  $\Omega \mathbf{k}$ , and  $\mathbf{v}$  whose components in the new coordinates become  $v'_x = v_y$  and  $v'_y = -v_x$ . Now consider the transformation to another new, doubly dashed, coordinate system defined by  $x'' = x$ ,  $y'' = y$  and  $z'' = -z$ . The reader will notice that the new system is left handed. The matrix  $a_{ij}^{(2)}$  for this transformation is

$$a^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (34)$$

Clearly, this transformation causes the  $z$  component of any vector to change sign, since anything pointing along the old positive  $z$  axis must now point along the negative  $z''$  axis. Suppose that this applies to the quantity  $\boldsymbol{\omega}$ . The transformation  $\boldsymbol{\omega}' = -\boldsymbol{\omega} = -\Omega \mathbf{k}$  would lead to a difficulty, because the velocity components for  $\mathbf{v}$  would become

$$v'_x = -\omega'_y = \Omega y \quad \text{and} \quad v'_y = \omega'_x = -\Omega x.$$

Thus the  $x$  and  $y$  velocities are now the negative of what they were before, although those axes were unchanged. The only way to correct this is to introduce a negative sign somewhere in the chain of definitions, and the standard place is the transformation law for  $\boldsymbol{\omega}$ . Thus the equation  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$  is kept intact (as is the definition of cross product in terms of components) as is the fact that  $\mathbf{v}$  and  $\mathbf{r}$  transform according to the

vector law. Nonetheless,  $\boldsymbol{\omega}$  transforms according to a pseudovector law, namely  $\omega'_i = a_{ij}^{(1)}\omega_j$ , but  $\omega''_i = -a_{ij}^{(2)}\omega_j$ . This is the difference between ordinary, or polar, vectors such as velocity, force and position, and axial vectors such as angular velocity, and angular momentum.

The difference between vector types





every point is doubled also. More can be seen, however, if the equation is rewritten using the results of the previous section.

$$\mathbf{u} = \left[ \mathbf{I} \left( \frac{3}{4r} + \frac{1}{4r^3} \right) + \frac{\mathbf{x}\mathbf{x}}{r^2} \left( \frac{3}{4r} - \frac{3}{4r^3} \right) \right] \cdot \mathbf{U} . \quad (39)$$

The new feature of this equation is the fact that the brackets do not contain  $\mathbf{U}$ ; the effects of the velocity of the sphere and the geometry of the sphere have been separated in the presentation of the equation.

To put it another way, in one dimension a scalar quantity  $\alpha$  depends linearly on a scalar quantity  $\beta$  when  $\alpha = k\beta$ , and moreover the coefficient of proportionality  $k$  is also scalar. In (39) there is a similar linear relationship  $\mathbf{u} = \mathbf{K} \cdot \mathbf{U}$ , only now the linking coefficient has become a tensor, because  $\mathbf{u}$  and  $\mathbf{U}$  are not in the same direction (as a scalar coefficient would imply).

A second example is provided by the drag on a non-spherical particle. The force  $\mathbf{F}$  is linearly related to the velocity of the particle  $\mathbf{U}$ , but not necessarily in the same direction, as is shown by writing  $\mathbf{F} = \mathbf{A} \cdot \mathbf{U}$ . The tensor  $\mathbf{A}$  is called a resistance tensor and it contains only geometrical factors and information about the particle shape, but nothing about the velocity. This is some help, but  $\mathbf{A}$  still contains 6 scalar components (not 9, because the reciprocal theorem shows it is symmetric), and could be a difficult object to work with. If, however, the particle is axisymmetric,  $\mathbf{A}$  can be simplified further. Suppose the axis of symmetry is  $\mathbf{d}$  (a unit vector), then

$$\mathbf{A} = K_1 \mathbf{d}\mathbf{d} + K_2 \mathbf{I} , \quad (40)$$

here  $K_1$  and  $K_2$  are scalars based on the shape. The proof first extracts the component of velocity parallel to  $\mathbf{d}$ . From above, this is  $\mathbf{U} \cdot \mathbf{d}\mathbf{d}$ . By symmetry, the force is parallel to this component, so  $\mathbf{F} =$

linear in these quantities (cf. equation (2) above), the force is a l

most general form for  $G_{ijk}$  is then

$$G_{ijk} = K_4 d_i d_j d_k + K_5 d_i \delta_{jk} + K_6 d_j \delta_{ik} + K_7 d_k \delta_{ij} . \quad (43)$$

Since  $E_{jk}$  is traceless, the term  $K_5 d_i \delta_{jk}$  cannot contribute to  $F_i$  and may therefore be dropped. Similarly, because  $E_{jk}$  is symmetric, the terms  $K_6 d_j \delta_{ik}$  and  $K_7 d_k \delta_{ij}$  will always contribute to  $F_i$  in the form  $K_6 + K_7$ , so one term could be dropped. However, for reasons of aesthetics, they are more commonly set equal. Thus  $G_{ijk}$  is reduced to depending on two independent scalar functions. As in the case of  $\mathbf{A}$ , there is some arbitrariness in assigning the two functions, cf. equations (40) and (41), and different authors might vary in their choices.

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